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ABOUT NEW CONDITIONS OF SOLVABILITY OF CHAPLYGIN'S PROBLEM FOR INTEGRO-SUM INEQUALITIES

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Анотація. В статті наведено нові умови розв'язуваності задачі Чаплигіна для розривних функцій в одновимірному та в багатовимірному випадках. Використовуючи результати Азбєлєва, Цалюка, робота продовжує дослідження інтегро-сумарних нерівностей Агарвала, Акінлі, Борисенка, Іоване, Лакшмікантама, Самойленка .

Аннотация. В статье приведены новые условия решаемости задачи Чаплыгина для разрывных функций в одномерном и в многомерном случаях. Используя результаты Азбелева, Цалюка, работа продолжает исследования интегро-суммарных неравенств Агарвала, Акинли, Борисенка, Иоване, Лакшмикантама, Самойленка.

Abstract. In the present work we sow some new conditions of the solvability of Chaplygin's problem for discontinuous functions which satisfy some integro-sum inequalities in one-dimensional and also in multidimensional case. We generalize some of the last results of the investigations of integro-sum inequalities in present of impulsive source by Agarwal, Akinyele, Borysenko, Iovane, Lakshmikantham, Samoilenko by using the results of Azbelev, Tsalyuk.

Key words: integro-sum inequalities, integro-sum equations, differential equations

INTRODUCTION

One of the central points in qualitative analysis of the impulsive differential is to find the estimates of the solutions of the system. In this case the basic roles is played by integral operator and differential inequalities in comparison method (Chaplygin's theorems). The applications of such theorems in qualitative methods are enough variously. For example, one of general approaches for searching trivial solution of the system $x' = f(x,t)$, where $x = \{x^i\}$— n-dimensional vector, is described like follow. Let us define some functional $\varphi(x,t)$ (for example, norma $\|x\|$, or Lyapounov's function $V(x,t)$).

On the solution y of the system $x' = f(x,t)$ let the functional $\varphi(x,t)$ satisfy the inequalities $P(\varphi) \le 0$, with P- some operator in semi organized space (for example, P is operator-kind, integral-kind or differential-kind. For equation $P(u) = 0$ if Chaplygin's theorem is true (i.e. operator inequality $P(z) \le 0$ define the inequality $z \le u$ between element z and the solution of u of the equation $P(u) = 0$), then $\varphi(y,t) \le u$). Farther, the functional $\varphi(x,t)$ is chosen so that the property of u follows from the property of y . Then the inequality $\varphi(y,t) \le u$ determines some criteria of stability. For example, in the investigations by R. Bellman [1] the norma was taken as $\varphi(x,t)$, and $P(u) = 0$ — is some linear integral equation

$$u(t) = c + \int_0^t \nu(s)u(s)ds\,, \tag{1}$$

for which the theorem about integral inequality (Gronwall's lemma) is true; for Lyapounov's method the equation $P(u) = 0$ played the role by some linear differential equation of the first order.

Similar schemas are used for research the existence, the unique, the correctness, the continuous dependence of parameters of the solutions. The main problem, which bound by using such schemas, connected with functional $\varphi(x,t)$, must be chosen by using the property of the equation $P(u) = 0$, for which the conditions of the theorem of Chaplygin are true.

Note the investigations of Nemytskiy, Krasnoselsky, Krein, Warewsky were devoted to this problem; our results are based on the investigations in [1-15].

CHAPLYGIN'S PROBLEM FOR INTEGRAL INEQUALITIES

The rezolvent of equation

$$x(t) = \int_a^t G(t,s)x(s)ds + \psi(t)\,, \tag{2}$$

with a continuous kernel $G(t,s)$ is nonnegative, if only $G(t,s) \geq 0$. Therefore, from the integral inequality

$$z(t) \geq \int_a^t G(t,s)z(s)ds + \psi(t) \tag{3}$$

It follows the estimate $z \geq u$, if the kernel nonnegative and u is the solution of the equation (2).

By using the conditions of Adamar lemma for necessary smoothness of function $K(t,s,x)$ for every pair of continuous functions $z(t)$, $u(t)$, it exists continuous function $G_{z,u}(t,s)$ such, that:

$$K(t,s,z(s)) - K(t,s,u(s)) = G_{z,u}(t,s)(z(s) - u(s)). \tag{4}$$

Assume that smoothness of $K(t,s,x)$ and its monotonously on x $\left(\frac{\partial}{\partial x}K(t,s,x) \geq 0\right)$ in integral inequality

$$z(t) \geq \int_a^t K(t,s,z(s))ds + \psi(t) \tag{5}$$

Guarantee the estimate $z \geq u$, where u is solution of the equation

$$x(t) = \int_a^t K(t,s,x(s))ds + \psi(t) \tag{6}$$

Denote $z(t) - u(t) = \eta(t)$. Then

$$\eta(t) \geq \int_a^t G_{z,u}(t,s)\eta(s)ds \geq 0 \tag{7}$$

ABOUT CONDITIONS OF SOLVABILITY OF THE CHAPLYGIN'S PROBLEM FOR PIECEWISE CONTINUOUS FUNCTIONS

The solvability of the Chaplygin's problem [3] for an integral operator in the case of discontinuous functions is related to finding of such function $z(t)$ with discontinuities of 1-st kind in the points t_k: $0 \le a = t_0 < t_1 < \ldots, \lim_{i \to \infty} t_i = \infty$ which satisfy such type of integro-sum inequality

$$z(t) > \int_a^t K(t,s,z(s))ds + \psi(t) + \sum_{a<t_k<t} \theta(t,t_k)\mu_k(z(t_k - 0)) \tag{8}$$

take place $z(t) \ge u(t)$. Here $u(t)$ is the solution of the equation

$$x(t) = \int_a^t K(t,s,x(s))ds + \psi(t) + \sum_{a<t_k<t} \theta(t,t_k)\mu_k(x(t_k - 0)) \tag{9}$$

continuous on each of intervals, $[t_k, t_{k+1}[$, $k = 0,1,2,...,$ $\theta(t,t_k)$ -continuous nonnegative at the $t \ge a$ functions ($k = 0,1,2,...,$), vector-function

$K(t,s,x(s)) = \{K^1(t,s,x^1,...,x^n),...,K^n(t,s,x^1,...,x^n)\}$ and $\psi(t) = \{\psi^1(t),...,\psi^n(t)\}$

is determined in the domain D: $a \le s < t < b$ $\|x\| < c$ ($b \le \infty, c \le \infty$), functions $\mu_k(z)$ - continuous nonnegative and nondecreasing by z .

1.Let vector-function $K(t,s,x)$ in D satisfies conditions of Karateodori:

a) $K(t,s,x)$ continuous by x for all t and almost all s and measurable by s for all t and x.

b) For arbitrary positive number $\gamma < c$ there are exist such functions $\mu_\gamma(t,s)$ and $\nu_\gamma(t_1,t,s)$ $(a \le s \le t \le t_1 < b)$ summed by s on an interval $[a,t]$, that

$$\sup_{\|x\| \le \gamma} \|K(t,s,x)\| \le \mu_\gamma(t,s) \text{ and } \sup_{\|x\| \le \gamma} \|K(t_1,s,x) - K(t,s,x)\| \le \nu_\gamma(t_1,t,s)$$

$$\lim_{t_1 - t \to +0} \left[\int_t^{t_1} \mu_\gamma(t_1,s)ds + \int_a^t \nu_\gamma(t_1,t,s)ds \right] = 0 \tag{10}$$

at fixed t or t_1.

2. K nondecreasing by x, for all t, almost all s .Then the following theorem is valid.

Theorem 1. Let $u_d(t)$ - continuous on each of intervals $[t_k, t_{k+1}[$, $k = 0,1,2,...,$, lower (upper) solution of equation.

$$x(t) = \int_a^t K(t,s,x(s))ds + \psi(t) + \sum_{a<t_k<t} \theta(t,t_k)\mu_k(x(t_k - 0)). \tag{11}$$

If vector-function $z(t)$ on $[a,d) \subset [a,b)$ satisfies to integral inequality

$$\varphi(t) = z(t) - \int_a^t K(t,s,z(s))ds - \psi(t) - \sum_{a<t_k<t} \theta(t,t_k)\mu_k(z(t_k - 0)) \ge 0 \tag{12}$$

($\varphi(t) \le 0$), where $\psi(t)$, $\theta(t,t_k)$ - continuous nonnegative for $t \ge a$ a function ($k = 0,1,2,...,$), except for $z(t)$, has the 1st kinds of discontinuities in the points t_k: $0 \le a = t_0 < t_1 < \ldots, \lim_{i \to \infty} t_i = \infty$, functions $\mu_k(z)$ - continuous nonnegative and non-decreasing by z , then $z(t) \ge u_d(t)$ ($z(t) \le u_d(t)$) at $t \in [a,d)$.

Proof. We will consider an interval $[a,t_1[$. Then equation (3.1) and inequality (3.2) will have a kind accordingly:

$$x(t)=\int_a^t K(t,s,x(s))ds+\psi(t) \tag{13}$$

$$\varphi(t)=z(t)-\int_a^t K(t,s,z(s))ds-\psi(t)\geq 0 \tag{14}$$

As functions in (4) and (5) on an interval $[a,t_1[$ continuous, following [4], we have: $z(t)\geq u_d(t)$, at all $t\in[a,t_1[$.

We will consider an interval $[t_1,t_2[$. Then equation (2) and inequality (3) will acquire a kind:

$$x(t)=\int_a^t K(t,s,x(s))ds+\psi(t)+\theta(t,t_1)\mu_1(x(t_1-0)) \tag{15}$$

$$\phi(t)=z(t)-\int_a^t K(t,s,z(s))ds-\psi(t)-\theta(t,t_1)\mu_1(z(t_1-0))\geq 0 \tag{16}$$

We will convert (3.6):

$$z(t)\geq\int_a^t K(t,s,z(s))ds+\psi(t)+\theta(t,t_1)\mu_1(z(t_1-0)) \tag{17}$$

We will consider a difference between (6) and (8). We have:

$$z(t)-x(t)\geq\int_a^t\left[K(t,s,z(s))-K(t,s,x(s))\right]ds+\theta(t,t_1)\left[\mu_1(z(t_1-0))-\mu_1(x(t_1-0))\right] \tag{18}$$

As after the condition of theorem $\theta(t,t_k)\geq 0\ \ \forall a<t_k<t, k=1,2,...$ then $x(t_1-0)\leq z(t_1-0)$.

We will consider expression:

$$\int_a^t\left[K(t,s,z(s))-K(t,s,x(s))\right]ds=\int_a^{t_1}\left[K(t,s,z(s))-K(t,s,x(s))\right]ds+\int_{t_1}^t\left[K(t,s,z(s))-K(t,s,x(s))\right]ds \tag{19}$$

$t\leq t_2$. The $K(t,s,z(s))\geq K(t,s,x(s)), \forall s\in[a,t_1[$, as $z(s)\geq x(s), \forall s\in[a,t_1[$ and this inequality is carried out for arbitrary fixed $s\leq t, t\in[a,t_1[$. Like there is justice of inequality $K(t,s,z(s))\geq K(t,s,x(s))$ and for an interval $[t_1,t_2[$. Thus $z(t)\geq u_d(t)$ we will get estimation, applying the method of mathematical induction on each of intervals of continuity of function $[t_k,t_{k+1}[$, taking into account finite jump in the points of break $\{t_k\}$ and monotonous of functions in inequality. A theorem is proved.

THE PROBLEM OF REDUTION FUNCTIONAL INTEGRAL INEQUALITIES FOR FUNCTION OF SEVERAL VARIABLES TO ONE-DIMENSIONAL INEQUALITIES

We propose a constructive approach to the problem to reducing such inequalities to one-dimensional.

This gives a generalization of the Bihari - Rahmatullina - Akinyele result.

Let's consider Euklidean space R^n with points $x=\left(x^1,...,x^n\right)$, $x^0=\left(x^{01},...,x^{0^n}\right)$, so that $x^0 \le x(x^{0^i} \le x^i)$ $\forall i=\overline{1,n}.$ Denote

$$\int\limits_{x^0}^{x} ...du = \int\limits_{x^{01}}^{x^1} ... \int\limits_{x^{0^n}}^{x^n} ...du_1...du_n \tag{20}$$

$$\sum_{x^0<x_k<x} \alpha_k = \sum_{x^0<x_{k_1}<x^1,...,x^{0^n}<x_{k_n}<x^n} \alpha_k \tag{21}$$

Introduce the space F of continuous functions $f: R^n \to R^n$ such that

A) $f(x)=\left(f_1(x), f_2(x),..., f_n(x)\right)$, **where** $f_j: R^n \to R$, $j=1,2,...,n$;

B) $f(x) \le x$;

C) $\lim\limits_{|x|\to\infty} f_j(x)=\infty$, $j=1,2,...,n$.

Assume that it is given a multi-dimensional integro-sum inequality of the type

$$u(t_1,...,t_n) < f(t_1,...,t_n) + \int\limits_{c_1}^{t_1} ... \int\limits_{c_n}^{t_n} K\left[t_1,...,t_n,s_1,...,s_n,u\left(p(s_1,...,s_n)\right)\right]ds_1...ds_n + \\ + \sum_{c_1<\tau_i^{(1)}<t_1,...,c_n<\tau_i^{(n)}<t_n} \mu_k\left(t_1,...,t_n \tau_i^{(1)},...,\tau_i^{(n)}\right)\eta_i\left[u\left(q\left(\tau_i^{(1)}-0,...,\tau_i^{(n)}-0\right)\right)\right] \tag{22}$$

where $i=1,2,...,n$ if $t_j \in \left[c_j,\infty\right]$ and $i=\overline{1,k}, k<\infty$, if $t_j \in \left[c_j,T_j\right], T_j=const, j=\overline{1,n},$ i. e. the set $\left\{\tau_i^{(j)}, j=\overline{1,n}\right\}$ is finite in the bounded domain $\left[c,T\right]=\left\{\left[c_1,T_1\right],...,\left[c_n,T_n\right]\right\}$ and is countable in the infinite domain $p,q \in F$.

Suppose, that a set of vectors $x_i=\left\{\tau_i^{(1)},...,\tau_i^{(n)}\right\}$ is ordered $x_i < x_{i+1}$ $\forall i \in N$ $\tau_i^{(k)} < \tau_{i+1}^{(k)}, k=\overline{1,n}.$

We assumed that the function $u(t)$ has finite discontinuities $u\left(\tau_i^{(1)}-0,\tau_i^{(n)}-0\right) \ne u\left(\tau_i^{(1)}+0,\tau_i^{(n)}+0\right)$ at the points $\left(\tau_1^{(1)},...,\tau_1^{(n)}\right),\left(\tau_2^{(1)},...,\tau_2^{(n)}\right),...$

The following theorem holds.

Theorem 2. Let $u_d(t)$ - continuous on each of domens

$$D_k : D_k = D_{k_1,...,k_n} = \left\{t : t^1 \in \left[t_{k-1}, t_k\right[,..., t^n \in \left[t_{k_n-1}, t_{k_n}\right[, k_j = 1,2,...,n\right\},$$

lower (upper) solution of equation

$$x(t) = \int\limits_a^t K\left(t,s,x\left(p(s)\right)\right)ds + \psi(t) + \sum_{a<t_k<t} \theta(t,t_k)\mu_k\left(x\left(q(t_k-0)\right)\right) \tag{23}$$

If vector-function $z(t)$ satisfies to integral inequality

$$\varphi(t) = z(t) - \int\limits_a^t K\left(t,s,z\left(p(s)\right)\right)ds - \psi(t) - \sum_{a<t_k<t} \theta(t,t_k)\mu_k\left(z\left(q(t_k-0)\right)\right) \ge 0 \tag{24}$$

($\varphi(t) \le 0$) where $\psi(t)$, $\theta(t,t_k)$ continuous nonnegative for $t \ge a$ the functions ($k=0,1,2,...,n$), except for $z(t)$, which has the discontinuities in the points $\{t_k\} = \{t_{k_1},...,t_{k_n}\}$, $x(t_i - 0) \neq x(t_i + 0), \forall i \in N$, the functions $\mu_k(q(z))$ are continuous nonnegative and non-decreasing by $z, p \in F, q \in F,$ then

$$z(t) \geq u_d(t) \tag{25}$$

($z(t) \leq u_d(t)$) at $t \in [a, d)$.

Proof. By using the method of reducing a multi-dimensional inequality (9) to a one-dimensional inequality [10], we shall pass from the many-dimensional integration-summation equation (10) to the one-dimensional equation:

$$x^*(t) = \psi^*(t) + \int_a^t K^*(t, s, x^*(p(s)))ds + \sum_{a<t_k<t} \theta^*(t, t_k)\mu_k^*(x^*(q(t_k - 0))), \tag{26}$$

where $x^*(t)$ has 1-st kind discontinuities in the points t_k (x^* - piecewise continuous function)

Respectively, we write one-dimensional inequality (11) in the form

$$\varphi^*(t) = z^*(t) - \int_a^t K^*(t, s, z^*(p(s)))ds - \psi^*(t) - \sum_{a<t_k<t} \theta^*(t, t_k)\mu_k^*(z^*(q(t_k - 0))) > 0. \tag{27}$$

$(\varphi^*(t) < 0)$

Consider the interval $[a, t_1[$. Then the equation (13) and the inequality (14) are in the following form

$$x^*(t) = \psi^*(t) + \int_a^t K^*(t, s, x^*(p(s)))ds \tag{28}$$

$$\varphi^*(t) = z^*(t) - \int_a^t K^*(t, s, z^*(p(s)))ds - \psi^*(t) \geq 0. \tag{29}$$

Note that p, q are scalar functions of a scalar argument, $p, q \in F$.

Since the functions from (15), (16) are continuous on the interval $[a, t_1[$, by using [4] we have:

$$z^*(t^0) \geq u_\varphi^*(t^0) \tag{30}$$

at all $t \in [a, t_1[$.

Thus the inequality (17) holds in some neighbourhood of the point t^0 ; moreover, the inequality (17) will be true for an arbitrary point $t^* \in [a, t_1[$, i.e. $z^*(t) \geq u_\varphi^*(t)$ $\forall t \in [t^0, t_1[$. Consider the interval $[t_1, t_2[$ the equation (13) and inequality (14) are on this interval of the form

$$x^*(t) = \psi^*(t) + \int_a^t K^*(t, s, x^*(p(s)))ds + \theta^*(t, t_1)\mu_1^*(x^*(q(t_1 - 0))), \tag{31}$$

$$\varphi^*(t) = z^*(t) - \int_a^t K^*(t, s, z^*(p(s)))ds - \psi^*(t) - \theta^*(t, t_1)\mu_1^*(z^*(q(t_1 - 0))) > 0. \tag{32}$$

Having considered the difference of (18) and (19), we obtain

$$x^*(t)-z^*(t)\le\int_{t^0}^{t}\left[K^*\left(t,s,x^*(p(s))\right)-K^*\left(t,s,z^*(p(s))\right)\right]ds+$$
$$+\theta^*(t,t_1)\left[\mu_1\left(x^*(q(t_1-0))\right)-\mu_1\left(z^*(q(t_1-0))\right)\right].$$

Since by the condition of the theorem, $\theta^*(t,t_k)\ge 0\quad \forall a<t_k<t,\quad k=1,2,...,$ we have $x^*(t_1-0)\le z^*(t_1-0)$, $x^*(q(t_1-0))\le z^*(q(t_1-0))$. Consider the expression

$$\int_a^t\left[K^*\left(t,s,x^*(p(s))\right)-K^*\left(t,s,z^*(p(s))\right)\right]ds=$$
$$=\int_a^t\left[K^*\left(t,s,x^*(p(s))\right)-K^*\left(t,s,z^*(p(s))\right)\right]ds+\int_{t_1}^t\left[K^*\left(t,s,x^*(p(s))\right)-K^*\left(t,s,z^*(p(s))\right)\right]ds,$$

$t\le t_2$.

It is obvious that $K^*\left(t,s,x^*(p(s))\right)\le K^*\left(t,s,z^*(p(s))\right), \forall s\in[a,t_1[$. If $x^*(s)\le z^*(s)\forall s\in[a,t_1[$ and this inequality holds for arbitrary fixed $s\le t, t\in\left[t^0,t_1\right[$.

The proof of the inequality

$$K^*\left(t,s,x^*(p(s))\right)\le K^*\left(t,s,z^*(p(s))\right)$$

in the interval $[t_1,t_2[$ similar etc. Thus we obtain the estimate $z(t)\ge u_d(t)$ by using of mathematical induction on each interval $[t_k,t_{k+1}[$ of continuity of the functions x^*,z^*, taking into account the finiteness of jumps in the points of discontinuities $\{t_k\}$ and monotonic of the functions in the inequality. Then, we obtain the inequality (4.4) and the theorem is proved.

CONCLUSIONS

A. Suppose, that in theorem 3.1 $K(t,s,x)=p(t)x^m$ ($m>0$), $p(t):R^+\to R^+$, $\psi(t)$ - positive monotonously nondecreasing at $t\ge a$ function, $\theta(t,t_k)=\beta_k=const\ge 0$, $\mu_k=x(t_k-0)$, then

1.) $$z(t)\le u_d(t)=\psi(t)\prod_{a<t_i<t}(1+\beta_i)\left[1+(1-m)\int_a^t\psi^{m-1}(\tau)p(\tau)d\tau\right]^{\frac{1}{1-m}}$$, if $0<m<1$;

2.) $$z(t)\le u_d(t)=\psi(t)\prod_{a<t_i<t}(1+\beta_i)\exp\left(\int_a^t p(\tau)d\tau\right)$$, if $m=1$;

3.) $$z(t)\le u_d(t)=\psi(t)\prod_{a<t_i<t}(1+\beta_i)\left[1-(m-1)\prod_{t_0<t_i<t}(1+\beta_i)^{m-1}\int_{t_0}^t\psi^{m-1}(\tau)p(\tau)d\tau\right]^{-\frac{1}{1-m}},$$

$m>1$, $\forall t\ge a$: $\int_a^t\psi^{m-1}(\tau)p(\tau)d\tau<\dfrac{1}{(m-1)\prod_{a<t_i}(1+\beta_i)^{m-1}}$. (Theorem 3.1.2, p.176, [15]).

B. From the results of the theorem 3.1 the next statement follows. Suppose, that $K(t,s,x)=q(t)g(s)x^m(s)$. Then $q(t)\ge 1$, $\forall t\ge t_0$, $g(t): R^+\to R^+$, $\psi(t)=h(t)$. Then:

4.) $$z(t)\le u_d(t)=h(t)q(t)* \\ *\prod_{a<t_i<t}\left(1+\beta_i q(t_i)\right)\left[1+(1-m)\prod_{t_0<t_i<t}\left(1+\beta_i\right)^{m-1}\int_{t_0}^{t} g(s)q^m(s)n^{m-1}(s)ds\right]^{\frac{1}{1-m}}$$, if $0<m<1$, $\forall t\ge a$

5.) $$z(t)\le u_d(t)=h(t)q(t)\prod_{a<t_i<t}\left(1+\beta_i q(t_i)\right)\exp\left[\int_a^t g(s)q(s)ds\right]$$, if $m=1$

6.) $$z(t)\le u_d(t)=h(t)q(t)* \\ *\prod_{a<t_i<t}\left(1+\beta_i q(t_i)\right)\left[1-(m-1)\prod_{t_0<t_i<t}\left(1+\beta_i q(t_i)\right)^{m-1}\int_a^t g(s)q^m(s)n^{m-1}(s)ds\right]^{-\frac{1}{1-m}}$$

if $m>1$, $\forall t\ge a$: $\int_a^t g(s)q^m(s)n^{m-1}(s)ds<(m-1)^{-1}\prod_{a<t_i<t}\left(1+\beta_i q(t_i)\right)^{1-m}$ (Theorem 3.2.2, p.185, [15]).

C. Suppose, that in theorem 3.1 $K(t,s,x)=g(s)x(s)$, $g(t):R^+\to R^+$, $\psi(t)=h(t)$ - positive nondecreasing function for $t\ge a$, $\theta(t,t_i)=\beta_i=const\ge 0$, $\mu_k=x^m(t_k-0)$, parameter $m>0$. Then

7.) $$z(t)\le u_d(t)=h(t)\prod_{a<t_i<t}\left(1+\beta_i h^{m-1}(t_i)\right)\exp\left[\int_a^t g(s)ds\right]$$, if $0<m\le 1$, $\forall t\ge t_0$;

8.) $$z(t)\le u_d(t)=h(t)\prod_{a<t_i<t}\left(1+\beta_i h^{m-1}(t_i)\right)\exp\left[m\int_a^t g(s)ds\right]$$, if $m\ge 1$, $\forall t\ge t_0$.

(Theorem 2.1, p. 26 [11, see also proposition 2.10 (Borysenko D.S.), p.2148, [13]).

D. From the results of the theorem 4.1 the next statement follows (with assumption, that $K(t,s,x)=q(t)g(s)x^m(s)$, $q(t)\ge 1$ ($m>0$, $m\ne 1$) and $\mu_k=x^m(\tau(t)),\tau\in F, a=t^0$):

9.) $$z(t)\le u_d(t)=\psi(t)* \\ *\prod_{a<t_i<t}\left(1+\beta_i\psi^{m-1}(t_i)q^m(t_i)\right)\left[1+(1-m)\int_{t_0}^t g(s)\psi^{m-1}(s)q^m(\tau(s))\left[\frac{\psi(\tau(s))}{\psi(s)}\right]^m ds\right]^{\frac{1}{1-m}}$$, $\forall t\ge t_0$, $0<m<1$,

10.) $$z(t)\le u_d(t)=\psi(t)* \\ *\prod_{t_0<t_i<t}\left(1+\beta_i m\psi^{m-1}(t_i)q^m(t_i)\right)* \\ *\left\{1-(m-1)\left[\prod_{t_0<t_i<t}\left(1+\beta_i m\psi^{m-1}(t_i)q^m(t_i)\right)\right]^{m-1}*\int_{t_0}^t g(s)\psi^{m-1}(s)q^m(\tau(s))\left[\frac{\psi(\tau(s))}{\psi(s)}\right]^m ds\right\}^{-\frac{1}{m-1}}$$

$\forall t\ge t_0$: $\int_{t_0}^t g(s)\psi^{m-1}(s)q^m(\tau(s))\left[\frac{\psi(\tau(s))}{\psi(s)}\right]^m ds\le\frac{1}{m}$, $m>1$ $\prod_{t_0<t_i<t}\left(1+\beta_i m\psi^{m-1}(t_i)q^m(t_i)\right)<\left(1+\frac{1}{m-1}\right)^{\frac{1}{m-1}}$

(theorem 2.2, p.28, [2]).

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