UDC 512.534.5

V.D.Derech

ON FINITE SEMIGROUPS FOR WHICH THE INVERSE MONOID OF LOCAL AUTOMORPHISMS IS A CONGRUENCE-PERMUTABLE SEMIGROUP

Vinnytsia National Technical University

Анотація

В даній доповіді ми подаємо короткий огляд результатів щодо класифікації скінченних напівгруп, для яких інверсний моноїд локальних автоморфізмів є конгруенц-переставним.
Ключові слова: інверсна напівгрупа, інверсний моноїд локальних автоморфізмів, конгруенц-переставна напівгрупа.

Abstract

In the current report, we present a brief review of results about of the classification of a finite semigroups for which the inverse monoid of local automorphisms is a congruence-permutable semigroup.
Keywords: inverse semigroup, inverse monoid of local automorphisms, congruence-permutable semigroup.

1 Definitions and Terminology

Let S be an arbitrary semigroup. An element $e \in S$ is *idempotent* if $e^2 = e$. A semigroup every element of which is an idempotent is called a *band*. A commutative band is called a *semilattice*. A nontrivial semilattice is called *primitive* if each its nonzero element is an atom.

If there exists an element 1 of S such that for any x $x1 = 1x = x$, we say that 1 is an *identity* element of S and that S is *monoid* . We writs $S^1 = S$ if S is a monoid; otherwise S^1 is the monoid obtained from S by adjoining an identity element to S. We define $(a, b) \in R$ if $aS^1 = bS^1$, $(a, b) \in L$ if $S^1a = S^1b$ and $(a, b) \in J$ if $S^1aS^1 = S^1bS^1$. Further, $H = R \cap L$, $D = R \circ L$. It is well known that $D = R \circ L = L \circ R$.

A semigroup S is called *inverse* if, for any element $x \in S$, there exists a unique element x^{-1} such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. It is known (see [1]) that a semigroup is inverse if and only if it is regular and any two of its idempotents are commuting. Consider an arbitrary mathematical structure C. A local automorphism of the structure C is defined as an isomorphism between its substructures. The set of all local automorphisms with respect to ordinary operations of composition of binary relations forms *an inverse monoid of local automorphisms* of the mathematical structure C. We denote this monoid by LAut(C). The most natural example of inverse monoid is the monoid of all local automorphisms of a certain mathematical structure. For example, if C is a finite semigroup of right zeros, then $LAut(C)$ is a *symmetric inverse semigroup*. It is known that the inverse monoid $LAut(C)$ gives more information about the structure of C then the group of automorphisms of this structure.

The relation $\leq$ defined on any inverse semigroup S by $a \leq b \Leftrightarrow a = be$ for some $e \in E_S$ is the *natural partial order* on S. If $a \leq b$ and $c \in S$, then $ac \leq bc, ca \leq cb$ and $a^{-1} \leq b^{-1}$.

Let S be a semigroup. An equivalence relation θ on S is left (respectively right) congruence on S if for any $a, b \in S, (a, b) \in S$ implies $(ca, cb) \in \theta$ (respectively $(ac, bc) \in \theta$); θ is a *congruence* on S if it is both a left and a right congruence on S.

Let X be a partially ordered set. If the greatest lower bound (respectively least upper bound) of two elements a and b of X exists, we denote it by $a \wedge b$ (respectively $a \vee b$) and call this element the meet (respectively join) of a and b. If any two elements of X have a meet and a join, then X is a *lattice*. The set $Cong(S)$ of all congruences of a semigroup S forms a lattice under inclusion. A lattice $(L, \vee, \wedge)$ is *modular* if, for all elements a, b, c of L, the following identity holds $(a \wedge c) \vee (b \wedge c) = [(a \wedge c) \vee b] \wedge c$.

A semigroup is called *congruence-permutable* if, for any two of its congruences ω and σ, the equality $\omega \circ \sigma = \sigma \circ \omega$, where $\circ$ denotes the composition of binary relations, is true. A group is a classical example of congruence-permutable semigroup. Moreover, finite symmetric inverse semigroups, inverse monoids of local automorphisms of finite-dimensional vector spaces, inverse monoids of local automorphisms of finite linearly ordered semilattices, Brandt semigroups, and other semigroups are also congruence-permutable semigroups.

Let S be an arbitrary semigroup. By $Sub(S)$ we denote the lattice of all its subsemigroups. If the semigroup S contains the least nonempty subsemigroup (e.g., the identity subgroup of the group), then just this subsemigroup is regarded as the least element of $Sub(S)$. If the least nonempty subsemigroup in S does not exist, then we define the empty set as the least element of $Sub(S)$. In this case, the empty transformation is the null element of the inverse monoid $LAut(S)$. If $A \in Sub(S)$, then by ΔA we denote the relation of equality on the subsemigroup A. It is clear that ΔA is an idempotent of the monoid $LAut(S)$. Each idempotent of the semigroup $LAut(S)$ has the indicated form. If $A \in Sub(S)$, then by $h(A)$ we denote the height of the subsemigroup A in the lattice $Sub(S)$.

A semigroup S containing zero is called a *nilsemigroup* if, for any $x \in S$, there exists a natural number n such that $x^n = 0$.

For a prime number p, by $\mathbb{Z}_p$ denote the corresponding field. The set of all upper triangular matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$, where a, b, and c are arbitrary elements of the field $\mathbb{Z}_p$, forms a group with respect to the ordinary operation of multiplication, which is called a *Heisenberg group* over the field $\mathbb{Z}_p$ and denoted by $Heis(\mathbb{Z}_p)$.

In the present report we consider only finite semigroups.

2 Background. Formulation of the Required Results

The next two classical results is well known.

Theorem (Birkhoff). *If A is congruence-permutable algebra, then A is congruence-modular.*

Theorem (see e.g. [2]). *Every left congruence $\eta \subseteq R$ commutes with every right congruence $\theta \subseteq L$.*

Corollary. *Let Θ and Ω be congruences on a semigroup S. If $\Theta \subseteq H$ and $\Omega \subseteq H$ (where H is the Green's relation), then $\Theta \circ \Omega = \Omega \circ \Theta$.*

The following theorem yields important property about a congruence-permutable semigroup.

Theorem 1 (see [6], Theorem 4). *Let S be a congruence-permutable semigroup. Then the set of its ideals is linearly ordered with respect to the inclusion.*

The next theorem is a generalization of the two previous ones.

Theorem 2 (see [3], Theorem 1). *Suppose that S is an inverse semigroup with zero 0 whose semilattice of idempotents is of finite length. Any two congruences of the semigroup S are permutable if and only if its ideals are linearly ordered, and every congruence Θ has the form $\Theta = I \times I \cup \Omega$, where I is an ideal of the semigroup S, $\Omega \subseteq H$ and H is the Green relation.*

The following theorem provides the characterization of congruence-permutable inverse semigroups.

Theorem 3 (see [4], Theorem 2). *Let S be an inverse semigroup with zero whose semilattice E of idempotents has finite length. In this case, S is congruence-permutable if and only if the following two conditions are satisfied:*

(i) for any a and $b \in$, if $rank(a) = rank(b)$, then $SaS = SbS$;

(ii) for any $e \in E$ ($rank(e) \geq 2$), there exist idempotents f and w such that $f \neq w, f < e, w < e$, and $rank(f) = rank(w) = rank(e) - 1$.

Remark 1 (see [4], Theorem 1). *If the rank of an arbitrary element of the nontrivial inverse semigroup S with zero does not exceed 1, then the semigroup S is congruence-permutable if and only if it is a Brandt semigroup.*

Remark 2 (see [4], Theorem 2). *Note that condition (i) of Theorem 3 is equivalent to the linear ordering (with respect to inclusion) of the set of ideals of the semigroup S.*

The next theorem provides a criterion in order that the set of ideals of the inverse monoid $LAut(S)$ forms a chain under inclusion.

Theorem 4 (see [7], Theorem 1). *Let S be a finite semigroup. The ideals of the semigroup $LAut(S)$ are linearly ordered if and only if the nonisomorphic subsemigroups in the lattice $Sub(S)$ have different heights.*

3 The main classification theorems

Theorem 5 (see [5], Proposition 3). *Suppose that S is a finite semigroup. If the inverse monoid of local automorphisms $LAut(S)$ is congruence-permutable, then the semigroup S is either a group or a nilsemigroup, or a band.*

The next two theorems yields full list of finite bands and groups (respectively) for which the inverse monoid of local automorphisms is a congruence-permutable monoid.

Theorem 6 (see [7], Theorem 3). *Suppose that S is a finite band. An inverse monoid $LAut(S)$ is congruence-permutable only in the following case:*

(1) the band S is a linearly ordered semilattice;

(2) the band S is a primitive semilattice;

(3) the band S is a semigroup of right zeros;

(4) the band S is a semigroup of left zeros.

Theorem 7 (see [5], Theorem 2). *Suppose that G is a finite group. An inverse monoid $LAut(G)$ is congruence-permutable if and only if G is:*

(1) either an elementary Abelian p-group, where p is any prime number;

(2) or a Heisenberg group over the finite field $\mathbb{Z}_p$, where p is an arbitrary odd prime number.

We now give the description of finite nilsemigroups for which the inverse monoid of local automorphisms is a congruence-permutable monoid (see [8]). Among these semigroups, an especially important role is played by two nilsemigroups given by Tables 1 and 2 and denoted by K_1 and K_2, respectively.

$\star$	**0**	**a**	**x**	**y**
0	0	0	0	0
a	0	0	0	0
x	0	0	0	a
y	0	0	0	0

Table 1

$\star$	**0**	**a**	**b**	**x**	**y**
0	0	0	0	0	0
a	0	0	0	0	0
b	0	0	0	0	0
x	0	0	0	0	a
y	0	0	0	b	0

Table 2

We also especially mention the other two nilsemigroups given by Tables 3 and 4 and denoted by B_1 and B_2, respectively.

⋆	**0**	a	x	y	z
0	0	0	0	0	0
a	0	0	0	0	0
x	0	0	0	a	0
y	0	0	0	0	a
z	0	0	a	0	0

Table 3

⋆	**0**	a	b	x	y	z
0	0	0	0	0	0	0
a	0	0	0	0	0	0
b	0	0	0	0	0	0
x	0	0	0	0	a	b
y	0	0	0	b	0	a
z	0	0	0	a	b	0

Table 4

We also present three constructions of nilsemigroups for which the inverse monoid of local automorphisms is a congruence-permutable monoid (see [8].

Construction 0

We fix a two-element set $\{0, a\}$. Let a finite set X be such that $\{0, a\} \cap X = \emptyset$ and $|X| \geq 2$. We defined a binary operation $*$ on the set $\{0, a\} \cup X$ as follows:
$0 * y = y * 0 = 0$ for any $y \in \{0, a\} \cup X$
$a * y = y * a = 0$ for any $y \in \{0, a\} \cup X$,
if $x_k, x_m \in X$ and $x_k \neq x_m$, then $x_k * x_m = a$,
$z^2 = 0$ for any $z \in \{0, a\} \cup X$.

Construction 1

We fix a two-element set $\{0, a\}$. Assume that a finite set X is such that $\{0, a\} \cap X = \emptyset$ and $|X| \geq 3$. In X, we introduce a strict linear ordering $<$ and define a binary operation on the set $\{0, a\} \cup X$ as follows:
$0 * y = y * 0 = 0$ for any $y \in \{0, a\} \cup X$,
$a * y = y * a = 0$ for any $y \in \{0, a\} \cup X$,
if $x_k, x_m \in X$ and $x_k < x_m$, then $x_k * x_m = 0$ and $x_m * x_k = a$,
$z^2 = 0$ for any $z \in \{0, a\} \cup X$.

Construction 2

We fix a three-element set $\{0, a, b\}$. Assume that a finite set X is such that $\{0, a, b\} \cap X = \emptyset$ and $|X| \geqslant 3$. We introduce a strict linear ordering $<$ on X and define a binary operation $*$ on the set $\{0, a, b\} \cup X$ as follows:
$0 * y = y * 0 = 0$ for any $y \in \{0, a, b\} \cup X$,
$a * y = y * a = 0$ for any $y \in \{0, a, b\} \cup X$,
$b * y = y * b = 0$ for any $y \in \{0, a, b\} \cup X$,
if $x_k, x_m \in X$ and $x_k < x_m$, then $x_k * x_m = a$ and $x_m * x_k = b$,
$z^2 = 0$ for any $z \in \{0, a, b\} \cup X$.

The following theorem is complete classification of a finite nilsemigroups for which the inverse monoid of local automorphisms is a congruence-permutable semigroup.

Theorem 8 (see [8], Theorem 5)**.** *Let S be a finite nilsemigroup. The inverse monoid $LAut(S)$ is congruence-permutable only in the following cases:*

(1) the nilsemigroup S is a semigroup with zero multiplication;

(2) the nilsemigroup S is isomorphic to K_1 (see table 1);

(3) the nilsemigroup S is isomorphic to K_2 (see table 2);

(4) the nilsemigroup S is isomorphic to B_1 (see table 3);

(5) the nilsemigroup S is isomorphic to B_2 (see table 4);

(6) the nilsemigroup S has the structure described in Construction 0*;*

(7) the nilsemigroup S has the structure described in Construction 1*;*

(8) the nilsemigroup S has the structure described in Construction 2*;*

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Volodymyr Derech — PhD in mathematics, associate professor, Department of Mathematics, Vinnytsia National Technical University, Vinnytsia, email: derechvd@gmail.com