CLASSIFICATION OF FINITE SEMIGROUPS FOR WHICH THE INVERSE MONOID OF LOCAL AUTOMORPHISMS IS A Δ-SEMIGROUP

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ABSTRACT. A semigroup S is called a Δ-semigroup if the lattice of its congruences forms a chain relative to the inclusion. A local automorphism of the semigroup S is defined as an isomorphism between its two subsemigroups. The set of all local automorphisms of the semigroup S relative to the operation of composition forms an inverse monoid of local automorphisms. In the current paper we present a classification of all finite semigroups for which the inverse monoid of local automorphisms is a Δ-semigroup.
Keywords:

A local automorphism of the semigroup S is defined as an isomorphism between two subsemigroups of this semigroup. The set of all local automorphisms of the semigroup S with respect to the ordinary operation of composition of binary relations forms an inverse monoid of local automorphisms. We denote this monoid by $LAut(S)$. Next, a semigroup S is called congruence-permutable if $\xi \circ \eta = \eta \circ \xi$ for any pair of congruences ξ, η on S. A semigroup S is called a Δ-semigroup if the lattice of its congruences forms a chain relative to the inclusion. It is obvious that any Δ-semigroup is congruence-permutable. A semigroup each element of which is an idempotent is called a band. A semigroup S with zero is called a nilsemigroup if, for any $x \in S$, there exists a natural number n such that $x^n = 0$.

Theorem 1 (see [1], proposition 3)**.** *Let S be a finite semigroup. If the inverse monoid of local automorphisms $LAut(S)$ is a congruence-permutable, then the semigroup S is either a group or a nilsemigroup, or a band.*

Theorem 2. *Let S be a finite band or a finite nilsemigroup. The following statements are equivalent:*
(a) *$LAut(S)$ is a congruence-permutable inverse semigroup;*
(b) *$LAut(S)$ is a Δ-semigroup.*

The following theorem was proved in [2].

Theorem 3. *Let S be a finite band. The inverse monoid $LAut(S)$ is a congruence-permutable if and only if S is:*
(1) *either a linearly ordered semilattice;*
(2) *or a primitive semilattice;*
(3) *or a semigroup of right zeros;*
(4) *or a semigroup of left zeros.*

A finite nilsemigroups for which the inverse monoid of local automorphisms is a congruence-permutable semigroup describe in [3]. An especially important role is played by two nilsemigroups given by Table 1 and Table 2 and denoted by K_1 and K_2, respectively.

$\star$	**0**	**a**	**x**	**y**
0	0	0	0	0
a	0	0	0	0
x	0	0	0	a
y	0	0	0	0

Table 1

$\star$	**0**	**a**	**b**	**x**	**y**
0	0	0	0	0	0
a	0	0	0	0	0
b	0	0	0	0	0
x	0	0	0	0	a
y	0	0	0	b	0

Table 2

We also especially mention the other two nilsemigroups given by Table 3 and 4 and denoted by B_1 and B_2, respectively.

⋆	**0**	**a**	**x**	**y**	**z**
0	0	0	0	0	0
a	0	0	0	0	0
x	0	0	0	a	0
y	0	0	0	0	a
z	0	0	a	0	0

Table 3

⋆	**0**	**a**	**b**	**x**	**y**	**z**
0	0	0	0	0	0	0
a	0	0	0	0	0	0
b	0	0	0	0	0	0
x	0	0	0	0	a	b
y	0	0	0	b	0	a
z	0	0	0	a	b	0

Table 4

We now present three constructions used for the building of nilsemigroups for which the inverse monoid of local automorphisms is a delta-semigroup.

Construction 1

We fix a two-element set $\{0, a\}$. Let a finite set X be such that $\{0, a\} \cap X = \emptyset$ and $|X| \geq 2$. We defined a binary operation $*$ on the set $\{0, a\} \cup X$ as follows:

- $0 * y = y * 0 = 0$ for any $y \in \{0, a\} \cup X$;
- $a * y = y * a = 0$ for any $y \in \{0, a\} \cup X$;
- if $x_k, x_m \in X$ and $x_k \neq x_m$, then $x_k * x_m = a$;
- $z^2 = 0$ for any $z \in \{0, a\} \cup X$.

Construction 2

We fix a two-element set $\{0, a\}$. Assume that a finite set X is such that $\{0, a\} \cap X = \emptyset$ and $|X| \geq 3$. In X, we introduce a strict linear ordering $<$ and define a binary operation $*$ on the set $\{0, a\} \cup X$ as follows:

- $0 * y = y * 0 = 0$ for any $y \in \{0, a\} \cup X$;
- $a * y = y * a = 0$ for any $y \in \{0, a\} \cup X$;
- if $x_k, x_m \in X$ and $x_k < x_m$, then $x_k * x_m = 0$ and $x_m * x_k = a$;
- $z^2 = 0$ for any $z \in \{0, a\} \cup X$.

Construction 3

We fix a three-element set $\{0, a, b\}$. Assume that a finite set X is such that $\{0, a, b\} \cap X = \emptyset$ and $|X| \geq 3$. In X, we introduce a strict linear ordering $<$ and define a binary operation $*$ on the set $\{0, a, b\} \cup X$ as follows:

- $0 * y = y * 0 = 0$ for any $y \in \{0, a, b\} \cup X$;
- $a * y = y * a = 0$ for any $y \in \{0, a, b\} \cup X$;
- $b * y = y * b = 0$ for any $y \in \{0, a, b\} \cup X$;
- if $x_k, x_m \in X$ and $x_k < x_m$, then $x_k * x_m = a$ and $x_m * x_k = b$;
- $z^2 = 0$ for any $z \in \{0, a\} \cup X$.

Theorem 4 (see [3], Theorem 5)**.** *Let S be a finite nilsemigroup. The inverse monoid $LAut(S)$ is congruence-permutable only in the following cases:*

1. *the nilsemigroup S is a semigroup with zero multiplication;*
2. *the nilsemigroup S is isomorphic to K_1 (see table 1);*
3. *the nilsemigroup S is isomorphic to K_2 (see table 2);*
4. *the nilsemigroup S is isomorphic to B_1 (see table 3);*
5. *the nilsemigroup S is isomorphic to B_2 (see table 4);*
6. *the nilsemigroup S has the structure described in Construction 1;*
7. *the nilsemigroup S has the structure described in Construction 2;*
8. *the nilsemigroup S has the structure described in Construction 3;*

The next theorem yield full list of a finite groups for which the inverse monoid of local automorphisms is a Δ-semigroup.

Theorem 5. *Let G be a finite group. The inverse monoid $LAut(G)$ is a Δ-semigroup if and only if G is:*

1. *either a group of prime order p, where $p - 1 = 2^k$ for some nonnegative integer k;*
2. *or an elementary Abelian 2-group of order 2^n, where $n \geq 2$.*

Some combinatorial facts

Let $S = \{a, b, c\} \cup X$ be a semigroup, whose structure satisfy the construction 3. If $|X| = n$ and $n \geq 3$, then:

(1) $|LAut(S)| = 13n^2 + 12n + 5 + 2 \cdot \binom{2n}{n}$;
(2) $|E(LAut(S))| = 2^n + 3 \cdot (n+1)$;
(3) $|Con(LAut(S))| = 2n + 5$.

References

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