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CRITERION OF ABSOLUTE STABILITY OF SOLUTIONS OF STOCHASTIC DIFFUSION DYNAMIC INFORMATION SYSTEMS OF AUTOMATIC REGULATION WITH EXTERNAL DISTURBANCES

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Анотація. Одержано умови асимптотичної стійкості тривіального розв'язку стохастичних дифузійних динамічних систем автоматичного регулювання з зовнішніми збуреннями. В роботі проведено пошук критеріїв абсолютної стійкості розв'язків стохастичних дифузійних динамічних систем автоматичного регулювання (САР) із зовнішніми збуреннями. Проблема визначення достатніх алгебраїчних умов стійкості стохастичних дифузійних динамічних САР з післядією та врахуванням зовнішніх збурень є актуальною, оскільки її вирішення дає змогу досліджувати складні стохастичні системи за допомогою матричних розрахунків.
Ключові слова: абсолютна стійкість, інформаційна система, автоматичне регулювання, стохастичне рівняння.

Abstract. The conditions of asymptotic stability of the trivial solution of stochastic diffusion dynamic systems of automatic regulation with external disturbances are obtained. In work is to find criteria for the absolute stability of solutions of stochastic diffusion dynamic systems of automatic regulation (SAR) with external disturbances. The problem of determining sufficient algebraic conditions for the stability of stochastic diffusion dynamic SARs with aftereffect and taking into account external disturbances is relevant, since its solution makes it possible to study complex stochastic systems using matrix calculations.
Key Words: absolute stability, information system, automatic regulation, stochastic equation.

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INTRODUCTION

In order to research and study real-life random processes, a whole constellation of scientists-mathematicians study stochastic models, which are described by differential, differential functional, differential-difference equations with disturbances of various kinds, the parameters of which are random functions of time [1, 2,].

The main indicator of high-quality physical implementation of the process is its stability [4, 5].

As a result, the academician O.M. Lyapunov at the end of the 19th and the beginning of the 20th century created a new scientific trend in the theory of motion - the theory of stochastic stability, which has gained wide development [6, 7, 8].

The purpose of this work is to find criteria for the absolute stability of solutions of stochastic diffusion dynamic systems of automatic regulation (SAR) with external disturbances. The problem of determining sufficient algebraic conditions for the stability of stochastic diffusion dynamic SARs with aftereffect and taking into account external disturbances is relevant, since its solution makes it possible to study complex stochastic systems using matrix calculations.

1. STATEMENT OF THE PROBLEM

On a probabilistic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq t_0 \geq 0\}, \mathbb{P})$, the random process $x(t) \equiv x(t,\omega): [0,T] \times \Omega \to \square^n$is a solution of the nonlinear diffusion stochastic equation of automatic regulation [3] (NDSRAR):

$$dx(t) = \left[f_1\big(\xi_1(\omega)\big)Ax(t) + g\varphi(\sigma)\right]dt + f_2\big(\xi_2(\omega)\big)Bx(t)dw(t) \tag{1}$$

with initial conditions

$$x(t_0) = x_0 \in \square^n. \tag{2}$$

Here $\sigma = l^T x(t)$, $t \geq 0$, $\varphi(\cdot)$is a nonlinear differentiated function under the condition

$$\big(k\sigma - \varphi(\sigma)\big)\varphi(\sigma) > 0;\ \ k > 0;\ \ \varphi(0) = 0;\ \ \dot{\varphi}(\sigma) \geq 0, \tag{3}$$

that is, it $\varphi(\sigma)$is between straight lines

$$\varphi(\sigma) = 0;\ \ \varphi(\sigma) = k\sigma; \tag{4}$$

$g = (g_1, g_2, \ldots, g_n)^T \in \square^n$; $l = (l_1, l_2, \ldots, l_n)^T \in \mathbb{R}^n$; $\xi_i(\omega) \in \square^1$, $i = 1,2$, are known random variables with distribution laws $F_\xi(x) \equiv \square\{\omega: \xi_i(\omega) < x, x \in \square^1\}$ $f_i(\cdot)$, $i = 1,2$, – Berev continuous functions; $A \equiv \{a_{ij}\} \subset \mathbb{R}^1$, $B \equiv \{b_{ij}\} \subset \mathbb{R}^1$are constant real matrices of dimension $n \times n$.

Let the initial system of automatic adjustment [3]

$$dx(t) = \left[\mathbb{E}\big\{f_1\big(\xi_1(\omega)\big)\big\}Ax_1(t) + g\varphi(\sigma)\right]dt. \tag{5}$$

has one equilibrium state, and A and $A + kgl^T$are Hurwitz matrices; $\omega \in \Omega$.

2. CONDITIONS FOR THE ASYMPTOTIC STABILITY OF THE TRIVIAL SOLUTION OF STOCHASTIC DIFFUSION DYNAMIC SARS WITH EXTERNAL DISTURBANCES

Consider the Lyapunov-Krasovsky stochastic functional [1]:

$$v(x) = x^T H x + \int_0^{\sigma(x)} \varphi\,(y)dy, \tag{6}$$

where the symmetric positive definite matrix H is a solution of Sylvester's matrix equations [1]:

$$\mathbb{E}\{f_1(\xi_1(\omega))\}(A^T H + HA) + \mathbb{E}\{f_2^2(\xi_2(\omega))\}BHB = -I. \tag{7}$$

The $v(x)$inequality [1] is valid for:

$$v\big(\widetilde{H}\big)^2\big(\widetilde{H}\big)^2{}_{max_{min}} \tag{8}$$

Where

$$v\big(\widetilde{H}\big)\begin{cases} v\left(H + \frac{1}{2}\mathcal{X}kll^T\right) \text{ для}_{min_{min}} \\ v(H) \text{ для}_{min} \end{cases} \tag{9}$$

$$v\big(\widetilde{H}\big)\begin{cases} v\left(H + \frac{1}{2}\mathcal{X}kll^T\right) \text{ для}_{max_{max}} \\ v(H) \text{ для}_{max} \end{cases} \tag{10}$$

According to the substitution of Ito [2], the stochastic differential of the functional $dv(x)$at the solutions of NDSRAR (1) – (2) has the form:

$$dv(x) = dx^T(t)Hx(t)dt + x^T(t)Hdx(t) + f_2^2(\xi_2(\omega))x^T(t)B^THBx(t)dt +$$
$$+\mathcal{X}d\int_0^{\sigma(x)}\varphi(y)dy = x^T(t)[f_1(\xi_1(\omega))(A^TH + HA) + f_2^2(\xi_2(\omega))B^THB]x(t) + \mathcal{X}d\left\{\int_0^{\sigma(x)}\varphi(y)dy\right\} =$$
$$= x^T(t)[f_1(\xi_1(\omega))(A^TH + HA) + f_2^2(\xi_2(\omega))x^T(t)]x(t)dt + [\varphi(\sigma)g^THx(t) + x^T(t)Hg\varphi(\sigma)]dt+$$
$$+\mathcal{X}[\varphi(\sigma)f_1(\xi_1(\omega))l^TAx(t) + \varphi(\sigma)l^Tg\varphi(\sigma)]dt + \tfrac{1}{2}\mathcal{X}\dot{\varphi}(\sigma)(f_2(\xi_2(\omega))l^TBx(t))^2dt + [x^T(t)f_2(\xi_2(\omega))(B^TH +$$
$$+HB)x(t) + \mathcal{X}\dot{\varphi}(\sigma)f_2(\xi_2(\omega))l^TBx(t)]dw(t). \tag{11}$$

According to the definition of the stochastic differential, equality (11) should be understood as an integral equation, because it $\frac{dw(t,\omega)}{dt}$ does not exist with probability 1 [2].

Next, we will calculate the mathematical expectation on the left and right sides of the corresponding integral equality (11), taking into account the equality of zero from the Wiener-Ito integral [2] for $\forall t \in [0,T]$:

$$\mathbb{E}\left\{\int_0^t \Phi(t,\omega)dw(t,\omega)\right\} = 0.$$

As a result, we get:

$$\mathbb{E}\left\{\frac{dv}{dt}\bigg|_{(1)}\right\} = \mathbb{E}\{x^T(t)[f_1(\xi_1(\omega))(A^TH + HA) + f_2^2(\xi_2(\omega))]x(t) + \varphi(\sigma)[g^TH + \mathcal{X}f_1(\xi_1(\omega))l^TA]x(t) +$$
$$+x^T(t)Hg\varphi(\sigma) + \tfrac{1}{2}\mathcal{X}\dot{\varphi}(\sigma)f_2^2(\xi_2(\omega))x^T(t)B^Tll^TBx(t)\}. \tag{12}$$

Under condition (3), we have

$$l^Tx(t)\varphi(\sigma) - \frac{\varphi^2(\sigma)}{k} > 0,$$

then equality (12) will turn into inequality:

$$\mathbb{E}\left\{\frac{dv}{dt}\bigg|_{(1)}\right\} \le \mathbb{E}\left\{\frac{dv}{dt}\bigg|_{(1)}\right\} + \mathbb{E}\left\{l^Tx(t)\varphi(\sigma) - \frac{\varphi^2(\sigma)}{k}\right\} = \mathbb{E}\{x^T(t)[f_1(\xi_1(\omega))(A^TH + HA) +$$
$$+f_2^2(\xi_2(\omega))B^THB]x(t)\} + +\mathbb{E}\left\{\varphi^T(\sigma)\left[g^TH + \frac{1}{2}\mathcal{X}f_1(\xi_1(\omega))l^TA + \frac{1}{2}l^T\right]x(t)\right\} +$$
$$+\mathbb{E}\left\{\varphi(\sigma)\left[Hg + \frac{1}{2}\mathcal{X}f_1(\xi_1(\omega))A^Tl + \frac{1}{2}l\right]x^T(t)\right\} + \mathbb{E}\left\{\varphi^T(\sigma)\left[\mathcal{X}l^Tg - \frac{1}{k}\right]\varphi(\sigma) +\right.$$
$$\left.+\frac{1}{2}\mathcal{X}f_2^2(\xi_2(\omega))\dot{\varphi}(\sigma)x^T(t)B^Tll^TBx(t)\right\}.$$

The last inequality can be written in vector-matrix form, namely:

$$\mathbb{E}\left\{\frac{dv(x,\sigma)}{dt}\bigg|_{(1)}\right\} \le \mathbb{E}\{\tilde{x}^T(t)\tilde{C}\tilde{x}(t)\}, \tag{13}$$

where $\tilde{x}^T(t) \equiv \left(x(t)\varphi, \sqrt{\dot{\varphi}}x^T(t)\right)$,

$$\tilde{C} \equiv \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \tag{14}$$

with correspondingly defined elements:

$$c_{11} = f_1(\xi_1)[A^T H + HA] + f_2^2(\xi_2)B^T HB;\;\; c_{21} = c^T 12 = Hg + \frac{1}{2}\mathcal{X} f_1(\xi_1)A^T l + \frac{1}{2}l;$$
$$c_{13} = c_{31} = 0;\; c_{23} = c_{32} = 0; c_{22} = \frac{1}{k} - \mathcal{X} l^T g;\;\; c_{33} = \frac{1}{2}\mathcal{X} f_2^2(\xi_2)B^T ll^T B.$$

Mathematical expectation (13) is negative at the solutions $x(t) \equiv x(t,\omega)$of system (1) if and only if the symmetric matrix

$$\mathbb{E}\{f_1(\xi_1(\omega))\}(A^T H + HA) + \mathbb{E}\{f_2^2(\xi_2(\omega))\}B^T HB \tag{15}$$

is negative definite, and the block symmetric matrix is $\tilde{C}$underdefinite [1].

We will denote further $\tilde{C} \leq 0$. Matrix $\mathbb{E}\{f_1(\xi_1(\omega))\}(A^T H + HA) + \mathbb{E}\{f_2^2(\xi_2(\omega))\}B^T HB$ is negative definite if and only if the matrix $H$is a solution of the Sylvester matrix equation (7), in which there are mathematical expectations:

$$0 < \mathbb{E}\{f_1(\xi_1(\omega))\} \leq K_1 < \infty,$$
$$0 < \mathbb{E}\{f_2^2(\xi_2(\omega))\} \leq K_2 < \infty. \tag{16}$$

The matrix $\tilde{C}$(14) is ill-defined only in the case of ill-defined matrix-blocks standing on its main diagonal, namely, the matrix is $\frac{1}{2}\mathcal{X}\mathbb{E}\{f_2^2(\xi_2(\omega))\}B^T ll^T B$ill-defined if and only if the number $\mathcal{X} < 0$, the number $\mathcal{X} l^T g - \frac{1}{k} < 0$, which is equivalent to the condition:

$$l^T g > 0. \tag{17}$$

Thus, the under definiteness of the block matrix $\tilde{C}$(14) under the conditions $\mathcal{X} < 01$, (7) and (17) also requires the underdefiniteness of the following matrix

$$\tilde{C}_1 \equiv \begin{bmatrix} -I & Hg + \frac{1}{2}\mathcal{X}\mathbb{E}\{f_1(\xi_1(\omega))\}A^T l + \frac{1}{2}l \\ \left(Hg + \frac{1}{2}\mathcal{X}\mathbb{E}\{f_1(\xi_1(\omega))\}A^T l + \frac{1}{2}l\right)^T & \mathcal{X} l^T g - \frac{1}{k} \end{bmatrix} \leq O_{(n+1)\times(n+1)}. \tag{18}$$

requirement (17) means the density of the matrix $\mathbb{E}\{f_1(\xi_1(\omega))\}A + kgl^T$, which characterizes the exponential stability of the matrix A.

We write down the condition of positive definiteness of the Lyapunov function (6) on the linear characteristic of the Hurwitz angle $\varphi(\sigma) = k\sigma$:

$$V|_{\varphi(\sigma)=\sigma} = x^T Hx + \mathcal{X}\int_0^\sigma k\, ydy = x^T Hx + \frac{1}{2}\mathcal{X} ky^2|_{y=0}^{y=l^T x^2} = x^T\left(H + \frac{1}{2}\mathcal{X} kll^T\right)x > 0. \tag{19}$$

Hence the positive definiteness of the matrix follows

$$H + \frac{1}{2}\mathcal{X} kll^T > O_{(n+1)\times(n+1)}.$$

Multiplying this inequality on the left by l^T, and on the right by l, we obtain an equivalent inequality

$$l^T Hl + \frac{1}{2}\mathcal{X} k(l^T l)^2 > 0,$$

Where

$$-\frac{2l^T Hl}{k(l^T l)^2} < X < 0. \tag{20}$$

Thus, the condition of negative definiteness of mathematical expectation requires negative definiteness of the matrix

$$\mathbb{E}\{f_1(\xi_1(\omega))\}[A^T H + HA] + \mathbb{E}\{f_2^2(\xi_2(\omega))\}B^T HB < 0$$

(condition (7)) and the fulfillment of the inequality

$$\widetilde{det\, C_1} < 0.$$

Let's open this determinant by the last column

$$\left[Hg + \frac{1}{2}(\mathcal{X}\mathbb{E}\{f_1(\xi_1(\omega))\}A^T l + I)l\right]^T \times [\mathbb{E}\{f_1(\xi_1(\omega))\}(A^T H + HA) + \mathbb{E}\{f_2^2(\xi_2(\omega))\}B^T HB]^{-1} \times$$
$$\times \left[Hg + \frac{1}{2}(\mathcal{X}\mathbb{E}\{f_1(\xi_1(\omega))\}l + I)l\right] < Xl^T g - \frac{1}{k}. \quad (21)$$

Let's get the statement.

Theorem. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq t_0 \geq 0\}, \mathbb{P})$NDSRAR (1), (2) be given on a probabilistic basis. Then the equilibrium position of the $x(t) \equiv x(t,\omega) = 0$system (1), (2) is absolutely stable in the mean square, if the conditions are met

1) $\mathbb{E}\{f_1(\xi_1(\omega))\}A + \mathbb{E}\{f_1(\xi_1(\omega))\}kgl^T$Hurwitz matrix; $l^T g > 0$;

2) there is a positive definite solution $H$of the matrix equation (7);

$$K_1 > E\{f_1(\xi_1(\omega))\} > 0; \quad (22)$$

3) the matrix inequality (13) is fulfilled with the choice $\mathcal{X} < 0$under the condition (20);

4) inequalities (21) with (22) are satisfied.

Let exist $v(x,t) \in C(\mathbb{D})$, where$\mathbb{D} = \{x \in \square^n, t \geq t_0 | \sum_{i=1}^n |x_i| < M, M > 0\}$,
which satisfies the conditions

$$F_1(|x|) \leq v(x,t) \leq F_2(|x|); \quad (22)$$

$$\frac{dv(x,t)}{dt} \leq -F_3(|x|) \quad (23)$$

where $F_i(r) \in C([0,+\infty))$, $F_i(0) = 0$; $\lim_{r\to+\infty} F_i(r) = 0$, $i = \overline{1,3}$.

Then:

I) The trivial solution of system (4) is asymptotically stable according to Lyapunov.

II)

$$|x| \leq F_1^{-1}[S^{-1}((t-t_0), v(x(t_0),t_0))]. \quad (25)$$

In inequality (25), S is defined as follows [3]

$$S(v,v_0) \equiv \int_{t_0}^{t} \frac{dv}{F_3[F_2^{-1}(v)]} \leq -(t-t_0).$$

Verification of conditions (1) - 4) is effective when using modern computer technologies.

CONCLUSION

The conditions of asymptotic stability of the trivial solution of stochastic diffusion dynamic systems of automatic regulation with external disturbances are obtained. In work is to find criteria for the absolute stability of solutions of stochastic diffusion dynamic systems of automatic regulation (SAR) with external disturbances. The problem of determining sufficient algebraic conditions for the stability of stochastic diffusion dynamic SARs with aftereffect and taking into account external disturbances is relevant, since its solution makes it possible to study complex stochastic systems using matrix calculations.

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