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ON PERFECT CODES IN THE DUAL-PANCAKE GRAPHS AND COMPLEXITY OF CONGRUENCE-CLASSES IN REGULAR VARIETIES

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Abstract

Here we define dual-pancake graphs and consider perfect codes in dual-pancake graphs. We find the existence of perfect codes in the dual-pancake graphs and consider some properties of them. In the second part we investigate complexity of congruence classes of algebras in some varieties.

Аннотация

Мы определяем двойственные панкейк графы и рассматриваем совершенные коды на двойственных панкейк графах. Устанавливаем существование совершенных кодов на двойственных панкейк графах и рассматриваем некоторые их свойства. Во второй части мы исследуем сложность вычислений конгруэнц-классов алгебр в некоторых многообразиях.

An interconnection network is usually represented by a graph and it is well-known that the pancake graphs are widely used as models for interconnection networks [1]. An independent set C in a graph is a *perfect code* (or *efficient dominating set*) if each vertex not in C is adjacent to exactly one vertex in C. The perfect codes in Cayley graphs on the symmetric group Sym_n are widely investigated [2]; in particular, it was shown the existence of perfect codes in the pancake graphs (general information on pancake graphs may be found, for example, in [3]). It is also known that the perfect codes are used in broadcasting algorithms for multiple messages (see, e.g., [4]). Our aim is investigating the dual-pancake graphs for interconnection networks. We show the existence of perfect codes in the dual-pancake graphs. The *dual-pancake graph* D_n, $n \geq 3$, is the Cayley graph on the symmetric group Sym_n of $n!$ permutations $\pi := [\pi_1, \pi_2, \dots, \pi_n]$, where $\pi_i = \pi(i)$ for every $i = 1, \dots, n$, with the generating set $SR := \{r_i \in Sym_n \mid 1 \leq i \leq n\}$ of all suffix-reversals r_i reversing the order of any substring $[n-i+1, \dots, n-1, n]$, $1 < i \leq n$, of a permutation π when multiplied on the right, i.e.,

$$[\pi_1, \dots, \pi_{n-i}, \pi_{n-i+1}, \dots, \pi_{n-1}, \pi_n] r_i = [\pi_1, \dots, \pi_{n-i}, \pi_n, \pi_{n-1} \dots, \pi_{n-i+1}].$$

It is a connected vertex-transitive $(n-1)$-regular graph of order $n!$. The n-dimensional dual-pancake networks based on the dual-pancake graphs $D_n, n \geq 3$, will be useful in computer science as models for interconnection networks such that processors are labeled by permutations of length n, and two processors are connected when the label of one is obtained from the other by some suffix-reversal. Here we will give the full characterization of the perfect codes in the dual-pancake networks. We show that there are exactly n perfect codes $C_1, \dots, C_n$ in the dual-pancake networks $D_n, n \geq 3$, presented as $C_k = \{[\pi_1, \dots, \pi_{n-1}, k] \mid \pi_j \in \overline{1,n} \backslash \{k\}, 1 \leq j \leq n-1\}$ such that $|C_k| = (n-1)!$. The descriptions of all connections between permutations from C_k (for a fixed $k = 1, \dots, n$) are also given. The above-mentioned construction and results may be generalized to a more general situation. We can fix some selected string (sequence) of the indices and begin the reversal process for this fixed selection of the indices. Again all our results are still true in this (more general) situation. Further we also study the symmetry of D_n and determine its full automorphism group; we show that D_n, $n \geq 5$, is the regular representation of the symmetric group Sym_n (here we use some results of [Godsil, On the full automorphism group of a graph, *Combinatorica*, 1 (1981), 243-256]).

A.I.Mal'tsev in 1954 shows that a nonempty subset $C \subseteq A$ of an algebra $\mathbb{A} = (A, F)$ is a congruence-class on $\mathbb{A}$ iff $\tau(C) \cap C = \emptyset$ or $\tau(C) \subseteq C$ for every unary polynomial τ of $\mathbb{A}$. It is a very useful characterization, but in special classes it can be concretized. The notion of

k –transferability is very closely connected with the notion of regularity. Algebra $\mathbb{A}$ is *regular*, if $\theta_1 = \theta_2$ for every congruences θ_1, θ_2 having a common congruence-class; for example, all groups are regular, but in general semigroups are not regular (regular algebras are characterized by B.Czakany, G.Gratzer and R.Wille). We note also that the principal congruences of $\mathbb{A}$ have k –transferability property if for all $a, b, c \in A$ there exist elements $d_1, \dots, d_k \in A$ such that $\theta(a, b) = \theta(d_1, \dots, d_k)$; here $\theta(X)$ denotes the congruence generated by a set X. A variety V has k –transferability property for principal congruences (briefly k –tppc), if all algebras of V have this property. We have the following

Lemma. For every variety V and natural k t.f.a.e.:

(1) V has k –tppc;

(2) there are natural l, m, a map $\varphi: \{1, \dots, m\} \to \{1, \dots, k\}$ and ternary terms $g_1, \dots, g_k$ and quaternary terms $\{f_j^i\}_{j=1,\dots,l}^{i=1,\dots,m}$ in the language of V such that: $x = f_1^1(z, x, y, z)$,

$$f_j^i\left(g_{\varphi(i)}(x, y, z), x, y, z\right) = f_{j+1}^i(z, x, y, z), \quad i = 1, \dots, m, j = 1, \dots, l-1,$$
$$f_l^i\left(g_{\varphi(i)}(x, y, z), x, y, z\right) = f_1^{i+1}(z, x, y, z), \quad i = 1, \dots, m-1,$$
$$f_l^m\left(g_{\varphi(m)}(x, y, z), x, y, z\right) = y, \quad z = g_n(x, x, z), n = 1, \dots, k.$$

We have also the following

Theorem 1. Let V be a k –tppc and $g_1, \dots, g_k$ are ternary terms in above-mentioned lemma. Let $\mathbb{A} = (A, F) \in V$ and $\emptyset \neq C \subseteq A$. Then t.f.a.e.:

(1) C is a class of some congruence $\theta \in Con(\mathbb{A})$,

(2) (i) for every m –ary operation $f \in F$, all elements $a_j, b_j \in A$ and for every element $c \in C$ the implication is true: $\&_{i=1}^n g_i\left(a_j, b_j, c\right) \in C \Rightarrow \&_{i=1}^n g_i(f(a_1, \dots, a_m), f(b_1, \dots, b_m), c) \in C$;

(ii) if $a, b, d \in A$, then $\&_{i=1}^n \left(g_i(a, b, c) \in C \,\&\, g_i(b, d, c) \in C\right) \Rightarrow \&_{i=1}^n g_i(a, b, d) \in C$;

(iii) if $a \in A$, $c, d \in C$, then $g_i(d, c, c) \in C$ for all $i = 1, \dots, n$ and

$$\&_{i=1}^n g_i(a, c, c) \in C \Rightarrow a \in C.$$

A computational aspect of universal algebra is now being given attention. It is known that the problem of 'whether given subset of the algebra of congruence regular variety constitute a congruence class' is solvable in polynomial time. Roughly speaking the time complexity of an algorithm is a function $f: \mathbf{N} \to \mathbf{N}$, that every problem of size n can be solved in no more than $f(n)$ number of computational steps. The theory of computational complexity can be found in the book [5]. Algorithms of polynomial (time) complexity (when f is a polynomial) are considered practically used (useful) and the algorithms of higher complexity (e.g., exponential) are considered as practically unsuitable. In our case it was necessary to solve the problem: for an algebra $\mathbb{A} = (A, F) \in V$ and for a subset $C \subseteq A$, whether C a congruence class? We have proved

Theorem 2. Suppose that V has a k –tppc with explicit ternary terms $g_1, \dots, g_k$. Then the above-mentioned problem is solvable in polynomial time.

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