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INTERPRETATIONS OF EQUATIONAL SPECIFICATIONS OF ABSTRACT DATA TYPES

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Abstract

Here we define a notion of interpretation for equational specifications of abstract data types. Then we show that relative this interpretation (after standard factorization) the class of all varieties generated by abstract data types forms a complete lattice.

Аннотация

Мы определяем понятие интерпретации для эквациональных спецификаций абстрактных типов данных. Затем показывается, что относительно этой интерпретации (после стандартной факторизации) класс всех многообразий, порождѐнных абстрактными типами данных составляет некоторую полную решѐтку.

We recall some definitions; all non-defined notions may be found in [1]. A signature is a set Σ, whose elements are called operation symbols, together with a mapping $ar{:}\Sigma \to \mathbb{N}$, called arity function, assigned to each operation symbol its finite arity. A realization of an n-ary operation symbol f in a set A is an n-ary operation on A;$f : A^n \to A$. Given a signature Σ, a Σalgebra $\mathbb{A}$ is a pair $\mathbb{A} := (A, \Sigma^A)$, consisting of a set A, called the carrier of $\mathbb{A}$, and a family $\Sigma^A := (\sigma^A | \sigma \in \Sigma)$ of realizations σ^A of operation symbols $\sigma \in \Sigma$.

Let Σ be a signature and X be a set (of variables). A Σ-term over X is defined recursively:

(a) each $x \in X$ is a Σ-term over X, and

(b) if $\sigma \in \Sigma$ is an n-ary operation symbol and $t_1, \dots, t_n$ are Σ-terms over X, then $\sigma t_1, \dots, t_n$ is a Σ-term over X.

The set of all Σ-terms over X is denoted by $T_\Sigma(X)$; it can be made into an (term) algebra $\mathbb{T}_\Sigma(X)$, if the construction of a new term from n given terms and an n-ary operation symbol is thought of as an operation $\sigma^{T_\Sigma(X)}(t_1, \dots, t_n) := \sigma t_1, \dots, t_n$ for any n-ary operation symbol $\sigma \in \Sigma$ and all $t_1, \dots, t_n \in T_\Sigma(X)$.

Algebraic specification of abstract data types is based on representation of data types by so-called many-sorted algebras (many-sorted algebras were first introduced by Higgins in 1963). A specification is then simply a many-sorted signature together with axioms for the operations. Such specification determines a class of models consisting of all many-sorted algebras subject to the given axioms.

Let S be a set (of sorts). An S-indexed family $A = (A_s | s \in S)$ of sets is called an S-sorted set. All set-theoretic operations, like union, intersection, direct products, etc., are defined component-wise.

Given a set E of equations, we denote by $Mod(E)$ the class of all algebras which validate E. A class $\mathcal{K}$ of algebras is said to be an equational class if $\mathcal{K} = Mod(E)$ for some set E of equations. Birkhoff's variety theorem can be translated directly to the many-sorted case: an abstract class of many-sorted algebras is a variety if and only if it is an equational class.

Recall that an equational specification is a pair (Σ, E) consisting of an S-sorted signature Σ and a set E of Σ-equations. Let Σ be an S-sorted signature. It is easy to see that a Σ-algebra $\mathbb{A}$ has no proper subalgebras if and only if $\mathbb{A}$ is an image of the (ground) term algebra $\mathbb{T}_\Sigma := \mathbb{T}\Sigma(\emptyset)$. Also, a Σ-algebra $\mathbb{A}$ is typical for a specification $(\Sigma, \emptyset)$ if and only if the unique homomorphism from $\mathbb{T}_\Sigma$ to $\mathbb{A}$ is injective (here "$\mathbb{A}$ is typical" means that a ground equation is valid in $\mathbb{A}$ if and only if it is deducible from Σ). As a corollary we see that a Σ-algebra $\mathbb{A}$ has no proper subalgebras and typical for a specification $(\Sigma, \emptyset)$ if and only if $\mathbb{A} \cong \mathbb{T}_\Sigma$.

Definition. (a) For a specification (Σ, E) , a Σ-algebra $\mathbb{A}$ is called (Σ, E) -Peano algebra if and only if

(a) $\mathbb{A}$ has no proper subalgebras, and 2) $\mathbb{A}$ is typical for (Σ, E).

(b) The (initial) abstract data types of a specification (Σ, E) is the class

$$ADT(\Sigma, E) := \{\mathbb{A} \mid \mathbb{A} \text{ is } (\Sigma, E)\text{-Peano algebra}\}.$$

It is easy to see that $ADT(\Sigma, \emptyset) = \{\mathbb{T}_{\Sigma}\}$, moreover $ADT(\Sigma, E) = \{\mathbb{T}_{(\Sigma, E)}\}$, where $\mathbb{T}_{(\Sigma, E)}$ is a factor-algebra of $\mathbb{T}_{\Sigma}$.

Let $V := var(ADT(\Sigma, E))$ be the variety generated by $ADT(\Sigma, E)$. For varieties

$$V_1 := var(ADT(\Sigma_1, E_1)) \text{ and } V_2 := var(ADT(\Sigma_2, E_2))$$

V_1 is interpretable in V_2 if and only if every Σ_1 -operation can be realized as a Σ_2 -term and the semantics of V_2 is contained in the semantics of V_1.

Theorem. The relation of interpretability in the class of all varieties (for all $ADT(\Sigma, E)$) is a quasi-order relation and after standard factorization (relative mutual interpretability) this class became a complete lattice.

References:

1. Donald Sannella and Andrzej Tarlecki Foundations of Algebraic Specification and Formal Software Development. Springer-Verlag, 2012, xvi+581p.,- ISBN 978-3-642-17335-6.