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INVERSE MONOID OF LOCAL AUTOMORPHISMS OF FINITE HEISENBERG GROUP

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Abstract

A local automorphism of a semigroup S is defined as an isomorphism between two of its subsemigroups. The set of all local automorphisms of a semigroup S with respect to an ordinary operation of composition forms an inverse monoid, which is denoted by $LAut(S)$. In the current conference paper we formulate (without proofs) some statements concerning the inverse monoid $LAut(H)$, where H is a finite Heisenberg group.

Keywords: Heisenberg group, inverse semigroup, inverse monoid of local automorphisms, congruence-permutable semigroup.

Анотація

Локальним автоморфізмом напівгрупи S називають ізоморфізм між двома її піднапівгрупами. Множина усіх локальних автоморфізмів напівгрупи S відносно звичайної операції композиції утворює інверсний моноїд, який позначається через $LAut(S)$. В даній статті ми формулюємо (без доведень) деякі твердження щодо інверсного моноїда $LAut(H)$, де H – скінченна група Гайзенберга.

Ключові слова: група Гайзенберга, інверсна напівгрупа, інверсний моноїд локальних автоморфізмів, конгруенц-переставна напівгрупа.

Let S be an arbitrary semigroup. An element $e \in S$ is idempotent if $e^2 = e$. A semigroup every element of which is an idempotent is called a band. A commutative band is called a semilattice. Let E be a finite band. By $h(a)$ we denote the height of the element $a \in E$. The set $\{x \in E\colon x \leq a\}$ is denoted by $a{\downarrow}$.

A semigroup S is called inverse if, for any element x , there is a unique element x^{-1} such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. It is known (see, for example, [1]) that a semigroup is inverse if and only if it is regular and two its arbitrary idempotents commute. Let S be an inverse semigroup. The set of all idempotents of S form the semilattice $E(S)$. Next, let C be an arbitrary mathematical structure. A local automorphism of the mathematical structure C is defined as an isomorphism between its substructures. The set of all local automorphisms of the structure C with respect to an operation of composition forms an inverse monoid, which is denoted by $LAut(C)$.

We say that a semigroup S is a congruence-permutable semigroup (or briefly: permutable semigroup) if $\theta \circ \xi = \xi \circ \theta$ is satisfied for every congruences θ and ξ on S. A group is a classical example of congruence-permutable semigroup. Moreover, finite symmetric inverse semigroups, inverse monoids of local automorphisms of finite-dimensional vector spaces, inverse monoids of local automorphisms of finite linearly ordered semilattices, Brandt semigroups, and some other semigroups are also congruence- permutable semigroups.

Let S be an arbitrary semigroup. By $Sub(S)$ we denote the lattice of all its subsemigroups. If the semigroup S contains the least nonempty subsemigroup (e.g., the identity subgroup of the group), then just this subsemigroup is regarded as the least element of $Sub(S)$. If the least nonempty subsemigroup in S does not exist, then we define the empty set as the least element of $Sub(S)$. In this case, the empty transformation is the null element of the inverse monoid $LAut(S)$. If $A \in Sub(S)$, then by ΔA we denote the relation of equality on the subsemigroup A. It is clear that ΔA is an idempotent of the monoid $LAut(S)$. Each idempotent of the semigroup $LAut(S)$ has the indicated form. If $A \in Sub(S)$, then by $h(A)$ we denote the height of the subsemigroup A in the lattice $Sub(S)$.

For a prime number p, by $\mathbb{F}_p$ denote the corresponding field. The set of all upper triangular matrices of the form $\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right)$, where a, b, and c are arbitrary elements of the field $\mathbb{F}_p$, forms a group with respect to the ordinary operation of multiplication, which is called a **Heisenberg group** over the field $\mathbb{F}_p$ and denoted by $Heis(\mathbb{F}_p)$.

We say that a semigroup A from a certain class of semigroups Ξ is defined by the inverse monoid $LAut(A)$ if the condition $LAut(A) \cong LAut(B)$ for a semigroup $B \in \Xi$ implies that $A \cong B$.

Theorem 1. *Let $H = Heis(\mathbb{F}_p)$ be a Heisenberg group over the finite field $\mathbb{F}_p$, where p is an arbitrary odd prime number. The following statements hold in $LAut(H)$.*

(1) $|E(LAut(H))| = p^2 + 2p + 4$.

(2) $|LAut(H)| = 2p^6 + p^5 - 2p^4$.

Theorem 2 (see [2])**.** *The inverse monoid $LAut(H)$ is congruence-permutable semigroup.*

Theorem 3 (see [3])**.** *Let $H = Heis(\mathbb{F}_p)$ be a Heisenberg group over the finite field $\mathbb{F}_p$, where p is an arbitrary odd prime number. Since the inverse monoid $LAut(H)$ is congruence-permutable, then the following conditions on $Sub(H)$ are satisfied:*

1. *if $A, B \in Sub(H)$ and $h(A) = h(B)$, then $A{\downarrow} \cong B{\downarrow}$;*
2. *if $F \in Sub(H)$ and $h(F) \geq 2$, then exist $C, D \in Sub(H)$ such that $C \subset F$, $D \subset F$, $C \neq D$ and $h(C) = h(D) = h(F) - 1$.*

Theorem 4. *In the class of all finite semigroups, the group H is defined by the inverse monoid $LAut(H)$.*

References

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