Voronoï's Impact on Modern Science, Book 6, Volume 2
Proceedings of the Sixth International Conference on Analytic Number Theory and Spatial Tessellations

A CONSTRUCTION OF REAL FUNCTIONS DEFINED WITH QUANTITIES OF DIGITS OCCURRENCES IN THE ARGUMENT REPRESENTATION

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Abstract. In the present paper, we consider one type of real functions whose algebraic expressions contain quantities of digits occurrences up to every position in the s-adic argument representation. We find conditions under which these functions are well defined and investigate some properties of these functions.

Introduction

There are several approaches used to define real functions with complicated specific features. Among them, we focus our attention on the approach based on the idea of description of the relationships with digits of the argument and values (in some representation, e.g., s-adic). Many fractal continuous functions, namely, singular, nowhere differentiable, etc., can be defined in this way.

In the present paper, we analyze a construction of functions whose algebraic expression contains quantities of digits occurrences up to every position in the s-adic argument representation.

We use following notation: let $s \geq 2$ be a positive integer, let $x \in [0,1]$ and $x = \sum_{k=1}^{\infty} \frac{\alpha_k}{s^k} \equiv \Delta^s_{\alpha_1\alpha_2\ldots\alpha_k\ldots}$ be the s-adic representation of x ($\alpha_k \in \{0,1,\ldots,s-1\}$), and let $N_j(k,x)$ be the number of digits j among $\alpha_1,\ldots,\alpha_k$ of x (in the s-adic representation).

1. Definition

For given values a_j ($j = \overline{0, s-1}$), let us consider a function which maps x into the number

$$f(x) = \sum_{k=1}^{\infty} \left(\frac{\alpha_k}{s^k} \prod_{j=0}^{s-1} a_j^{N_j(k,x)} \right). \tag{1}$$

Received September 1, 2018.
2010 *Mathematics Subject Classification.* Primary 26A30.
Key words and phrases. self-affine functions, singular functions, fractal functions.

First of all, we find conditions for which this function is well defined. It is correct if for different representations of a s-adic rational number x, the corresponding values are equal.

Hence, let $\Delta^s_{\alpha_1\alpha_2\ldots\alpha_n(0)}$, $\alpha_n \neq 0$, be the representation of a s-adic rational number x and let $\Delta^s_{\alpha_1\alpha_2\ldots(\alpha_n-1)(s-1)\ldots}$ be another well-known representation of x (for the sake of convenience, we denote this representation by x'). It is necessary that $f(x) = f(x')$.

We have:

$$f(x) = \sum_{k=1}^{n-1}\left(\frac{\alpha_k}{s^k}\prod_{j=0}^{s-1} a_j^{N_j(k,x)}\right) + \frac{\alpha_n}{s^n}\prod_{j=0}^{s-1} a_j^{N_j(n,x)}.$$

and

$$f(x') = \sum_{k=1}^{n-1}\left(\frac{\alpha_k}{s^k}\prod_{j=0}^{s-1} a_j^{N_j(k,x)}\right) + \frac{\alpha_n - 1}{s^n}\prod_{j=0}^{s-1} a_j^{N_j(n,x')} + \sum_{k=n+1}^{\infty}\frac{s-1}{s^k}\prod_{j=0}^{s-1} a_j^{N_j(k,x')}.$$

But

$$N_j(n,x') = N_j(n,x) + (\alpha_n == j+1) - (\alpha_n == j)$$

for all $j \in \{0,1,\ldots,s-1\}$ and

$$N_j(k,x') = N_j(k,x) - (\alpha_n == j) + (k-n)(j == s-1) + (\alpha_n - 1 == j)$$

for all $j \in \{0,1,\ldots,s-1\}$ and $k > n$ (here, we use notation: $a == b$ is 1 if $a = b$, and 0, otherwise).

Thus, $f(x) = f(x')$ if and only if

$$\frac{\alpha_n}{s^n}\prod_{j=0}^{s-1} a_j^{N_j(n,x)} = \frac{\alpha_n - 1}{s^n}\prod_{j=0}^{s-1} a_j^{N_j(n,x')} + \sum_{k=n+1}^{\infty}\frac{s-1}{s^k}\prod_{j=0}^{s-1} a_j^{N_j(k,x')}.$$

The last identity can be rewritten as

$$\frac{1}{s^n}\prod_{j=0}^{s-1} a_j^{N_j(n,x)}\left(\alpha_n - (\alpha_n - 1)\cdot\prod_{j=0}^{s-1}\frac{a_j^{\alpha_n==j+1}}{a_j^{\alpha_n==j}}\right) =$$

$$= (s-1)\prod_{j=0}^{s-1} a_j^{N_j(n,x)} \cdot \sum_{k=n+1}^{\infty}\frac{1}{s^k}\prod_{j=0}^{s-1}\frac{a_j^{j==\alpha_n-1} a_j^{(k-n)(j==s-1)}}{a_j^{j==\alpha_n}}.$$

If we denote $\alpha_n = q$, then this is equivalent to

$$\frac{1}{s^n}\prod_{j=0}^{s-1} a_j^{N_j(n,x)}\left(q-(q-1)\cdot\frac{a_{q-1}}{a_q}\right)=$$
$$=(s-1)\prod_{j=0}^{s-1} a_j^{N_j(n,x)}\cdot\frac{a_{q-1}}{a_q a_{s-1}^n}\cdot\sum_{k=n+1}^{\infty}\left(\frac{a_{s-1}}{s}\right)^k.$$

Let $a_{s-1} < s$. Then the sum on the right-hand side of the last equality is $\frac{(a_{s-1})^{n+1}}{(s-a_{s-1})s^n}$. Thus, after simplifying this equality, we find:

$$a_{q-1} = \frac{q(s-a_{s-1})}{(s-1)a_{s-1}+(q-1)(s-a_{s-1})}a_q. \tag{2}$$

This recurrence relation must be true for all $q = \overline{1, s-1}$. Hence, if we choose some integer $s \geq 2$ and $a_{s-1} < s$, then we get the unique function with values as in (1).

Example 1. Let $a_{s-1} = 1$. Then it is easy to verify that all a_q are equal to 1 and $f(x)$ is the identity function.

Example 2. Let $s = 2$. Then relation (2) can be rewritten as $a_0 + a_1 = 2$. In the case $a_0 = \frac{1}{2}$, $a_1 = \frac{3}{2}$, we get the strictly increasing singular function, which was described in [1].

Example 3. Let $s = 3$ and $a_2 \equiv c \in (0, 3)$. Then expression (1) can be rewritten as

$$f(x) = \sum_{k=1}^{\infty}\frac{\alpha_k}{3^k}\left(\frac{(3-c)^2}{3+c}\right)^{N_0(k,x)}\left(\frac{2c(3-c)}{3+c}\right)^{N_1(k,x)}c^{N_2(k,x)}.$$

These functions appeared and some of their properties were described in [4].

2. Some Properties

Theorem 1. *The function $f(x)$ defined by (1) with condition (2) on every a_q has following properties:*

1) $f(0) = 0$, $f(1) = \frac{(s-1)a_{s-1}}{s-a_{s-1}}$.

2) for all $i \in \{0, \dots, s-1\}$:

$$f\left(\frac{x+i}{s}\right) = \frac{ia_i}{s} + \frac{a_i}{s}f(x).$$

3) if $a_{s-1} > 1$ then $f(x)$ is an increasing function.

Proof. The first part of theorem immediately follows from the definition of f if we set $x = \Delta^s_{(0)}$ and $x = \Delta^s_{(2)}$.

Let us prove the second part. Indeed, for every $i \in \{0, \dots, s-1\}$ and $x = \Delta^s_{\alpha_1\alpha_2\dots}$, we get

$$f\left(\frac{x+i}{s}\right) = f\left(\Delta^s_{i\alpha_1\alpha_2\dots}\right) = \frac{i}{s}a_i + \sum_{k=1}^{\infty}\left(\frac{\alpha_k}{s^{k+1}}\prod_{j=0}^{s-1} a_j^{N_j\left(k+1,\frac{x+i}{s}\right)}\right) =$$
$$= \frac{i}{s}a_i + \frac{a_i}{s}\sum_{k=1}^{\infty}\left(\frac{\alpha_k}{s^k}\prod_{j=0}^{s-1} a_j^{N_j(k,x)}\right) = \frac{ia_i}{s} + \frac{a_i}{s}f(x).$$

We also note that $\frac{ia_i}{s} = f\left(\frac{i}{s}\right)$.

3) The inequality $\frac{q(s-a_{s-1})}{(s-1)a_{s-1}+(q-1)(s-a_{s-1})} < 1$ is equivalent to the inequality $a_{s-1} > 1$ and is independent of q. Hence, if $a_{s-1} > 1$, then the sequence $a_{s-1}, a_{s-2}, \dots$ is decreasing (or $a_0, a_1, \dots$ is increasing).

Let $x_1 < x_2$. More precisely, $x_1 = \Delta^s_{\alpha_1\dots\alpha_n\alpha_{n+1}\dots}$, $x_2 = \Delta^s_{\alpha_1\dots\alpha_n\beta_{n+1}\beta_{n+2}\dots}$, and $\beta_{n+1} > \alpha_{n+1}$. Let $x^\star = \Delta^s_{\alpha_1\dots\alpha_n(s-1)}$ and $x^{\star\star} = \Delta^s_{\alpha_1\dots[\alpha_n+1](0)}$. Obviously, $x^\star = x^{\star\star}$. Since $a_0, \dots, a_{s-1}$ is increasing, we see that $f(x_1) \le f(x^\star)$ and $f(x^{\star\star}) \le f(x_2)$. Thus, $f(x_1) \le f(x_2)$ and $f(x)$ is increasing. □

As we can see from the second part of the previous theorem, our functions have certain self-affine properties. They look like self-affine functions in Kono's sense but there is a slight difference from the indicated functions (see also [5, 3]).

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