

TAYLOR SERIES AND MACLAURIN SERIES: HISTORY, DERIVATION, APPLICATION TO FUNCTION APPROXIMATION

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Анотація: У роботі розглянуто ряди Тейлора та Маклорена як засоби подання та апроксимації функцій у математичному аналізі. Наведено короткий історичний огляд розвитку степеневих розкладів, подано виведення загальної формули, проілюстровано приклади для основних елементарних функцій. Показано практичне застосування рядів для побудови поліноміальних наближень та зазначено обмеження методу.

Ключові слова: ряд Тейлора; ряд Маклорена; степеневий розклад; апроксимація; залишковий член; аналіз функцій; математичний аналіз.

Abstract: The paper considers Taylor and Maclaurin series as means of representing and approximating functions in mathematical analysis. A brief historical overview of the development of power series is given, a general formula is derived, and examples for basic elementary functions are illustrated. The practical application of series for constructing polynomial approximations is shown, and the limitations of the method are indicated.

Keywords: Taylor series; Maclaurin series; power series; approximation; remainder term; function analysis; mathematical analysis.

Introduction

Taylor and Maclaurin series are one of the key tools of mathematical analysis, allowing a function to be expanded into an infinite sum of power terms based on its derivatives at a given point. This approach makes it possible to calculate the values of functions that cannot be expressed by elementary formulas, as well as to study their behaviour near the point of expansion.

The purpose of this report is to consider the historical development of Taylor and Maclaurin series, the theoretical derivation of the general formula, and the analysis of their applications in function approximation problems.

Research results

Historical development:

The idea of representing functions as infinite sums originated in India in the 14th–15th centuries. At the Madhava school in Sangamagram (c. 1350–1425), series were found for trigonometric functions — sine, cosine, and arctangent — now known as ‘Madhava series.’

Later, in the 17th century, Scottish mathematician James Gregory considered the possibility of expanding arctangent into a power series. The general formula for the expansion first appeared in 1715 in the work of English mathematician Brook Taylor, who systematised the idea of approximating functions using derivatives. [1]

A special case, where the expansion is performed around the point $a = 0$, was studied in detail by Scottish scientist Colin Maclaurin, thanks to which the corresponding series was named after him.

A historical analysis of the development of Taylor's theorem is presented, in particular, in G. Gibson's work ‘Taylor's Theorem and Bernoulli's Theorem: A Historical Note’ (1920), which highlights the evolution of ideas from early analytical approaches to formal proof.

The most important functional role of Taylor and Maclaurin series is the approximation of transcendental functions by polynomials, which serves as the basis for numerical methods in applied sciences. This approach allows complex calculations to be replaced by a series of simple arithmetic operations, which is crucial in the age of computer technology.

Numerical analysis and computational techniques:

The use of Taylor's polynomial $P_n(x)$, which is the finite sum of the first n terms of an infinite series, is a standard method for implementing elementary functions (such as e^x , $\sin x$, $\ln(1+x)$) in software and hardware for computers and engineering calculators [3]. High accuracy of calculations is guaranteed by controlling the residual term $R_n(x)$ in Lagrange form, which provides a quantitative estimate of the approximation error and ensures that the calculation error does not exceed the specified tolerance in a given neighbourhood of the expansion point [4]. In addition, Taylor series are fundamental to the derivation of many numerical methods [5]. They underlie numerical differentiation algorithms, where function derivatives are replaced by finite differences, and are also necessary for constructing highly efficient methods for solving ordinary differential equations, in particular the Runge-Kutta method [5].

Applications in physics and engineering:

In physical and engineering models, Taylor series are used to simplify nonlinear equations by linearising functions in the vicinity of the equilibrium point or for small perturbations [6]. For example, when analysing small oscillations (as in a simple pendulum), whose equation of motion contains $\sin(\Theta)$, the approximation of the first non-zero term of the Maclaurin series is used: $\sin(\Theta) \approx \Theta$ for small angles Θ . This allows the nonlinear system to be transformed into a linear one, which is easily amenable to analytical solution [6]. In the theory of relativity for low speeds ($v \ll c$), kinetic energy E_k is expanded into a Maclaurin series: $E_k \approx \frac{1}{2}m_0v^2$. This demonstrates how relativistic effects are considered as small corrections to classical results [7]. In optics and acoustics, when analysing wave propagation, the refractive index or sound velocity functions are often expanded into Taylor series around a certain point, which greatly simplifies the solution of wave equations [7].

Analysis of functions and limits:

Taylor series are also a powerful analytical tool for studying the local behaviour of a function. They are used to resolve indeterminacies of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ when calculating limits. By replacing functions with their Maclaurin series, one can easily isolate the main part of the expression and calculate the limit, which is an effective alternative to L'Hôpital's rule [4]. In addition, analysis of the Taylor series coefficients allows us to determine the nature of the critical point a : if the first non-zero derivative $f^{(n)}(a)$ has an even order n , the function has a local extremum; if it is odd, then it has an inflection point [4].

Conclusions:

Taylor and Maclaurin series are one of the most powerful tools in mathematical analysis, allowing complex functions to be approximated by polynomials of arbitrarily high order. The historical development of these series demonstrates a gradual transition from empirical observations of trigonometric series in Madhava's school to the formal mathematical apparatus created by Taylor and further developed by Maclaurin.

The practical significance of the series lies in the possibility of facilitating calculations, constructing approximations in mathematical modelling, applying them in numerical methods, solving differential equations, and analysing physical processes.

Despite their universality, Taylor series have certain limitations, in particular their dependence on the region of convergence and the impossibility of correctly approximating some functions. However, in most mathematical and applied problems, they remain indispensable due to their combination of simplicity of calculation and high accuracy of approximations.

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