



Classification of finite structurally uniform nilsemigroups

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Abstract

A finite semigroup S is called structurally uniform if any two subsemigroup of S are isomorphic whose heights in the partially ordered set of all subsemigroups of S are equal. Note that this class contains the class of all finite semigroups for which the inverse monoid of local automorphisms is congruence permutable. In this paper, we present a classification of finite structurally uniform nilsemigroups.

Keywords Inverse monoid of local automorphisms · Structurally uniform semigroup · Nilsemigroup · Congruence permutable semigroup

1 Introduction

Let C be an arbitrary mathematical structure. A local automorphism of the mathematical structure C is defined as an isomorphism between its substructures. The set of all local automorphisms of the structure C forms an inverse monoid with respect to composition (of partial one-one maps) which is denoted by $\text{LAut}(C)$. The investigation of relationships between the mathematical structure C and the inverse monoid $\text{LAut}(C)$ is an important problem in the theory of inverse semigroups. Many articles are devoted to this topic (see, for example, [5–10, 12, 14]).

Let S be an arbitrary finite semigroup. Denote by $\text{Sub}(S)$ the ordered set of all subsemigroups of S . The heights of any $U, V \in \text{Sub}(S)$ are the same if $m(U) = m(V)$ where $m(U)$ is the maximum of the cardinalities of all descending chains in $\text{Sub}(S)$ whose greatest element is U .

A finite semigroup S is called *structurally uniform* if any two subsemigroups in it are isomorphic whose heights in the partially ordered set of all subsemigroups are the same. It is known [4] that the semigroup S is structurally uniform if and only if the

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Table 1 The Cayley table for A

$\star$	0	a	x	y	z
0	0	0	0	0	0
a	0	0	0	0	0
x	0	0	0	a	0
y	0	0	0	0	a
z	0	0	a	0	0

lattice of all ideals of the inverse monoid $\text{LAut}(S)$ is a chain. A congruence permutable semigroup is a semigroup whose congruences commute under composition. If $\text{LAut}(S)$ is congruence permutable for a finite semigroup S then S is known to be structurally uniform (see [4, 5]). For a prime number p , denote the p -element field by $\mathbb{F}_p$. The set of all upper triangular matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$, where a, b , and c are arbitrary elements of the field $\mathbb{F}_p$, forms a group with respect to the usual multiplication, which is called a Heisenberg group over the field $\mathbb{F}_p$. In [3], it is proved that a finite group is structurally uniform if and only if it is a cyclic group of order p^n or an elementary Abelian p -group, where p is any prime number, or it is the quaternion group Q_8 , or a Heisenberg group over $\mathbb{F}_p$, where p is an odd prime number. A finite band is structurally uniform if it is either a right zero semigroup, or a left zero semigroup, or a linearly ordered semilattice, or a primitive semilattice (see [4]). A complete list of commutative structurally uniform nilsemigroups is given in [3].

A semigroup S containing zero 0 is called a nilsemigroup if, for any $x \in S$, there exists a natural number n such that $x^n = 0$.

In this paper, we complete the classification of all finite structurally uniform nilsemigroups. The main result (Theorem 1) is formulated in Sect. 2, and it is proved in Sect. 3.

The basic concepts and results of the theory of semigroups and inverse semigroups can be found in monographs [1, 11] and [13], respectively.

2 Statement of the main result

In this section we formulate the main result of the paper, and for completeness, we give the semigroups $A, B, C, D(n), F(n), G(n), H(n), K(n), R(n), T(n)$ (n is a positive integer) which appear in it. Semigroups A, B, C are given via their Cayley tables as follows (Tables 1, 2, 3):

Next, we fix a positive integer n , an n -element set X and a three-element set $\{0, a, b\}$ such that $\{0, a, b\} \cap X = \emptyset$. Each of $D(n), F(n), G(n), H(n), K(n), R(n), T(n)$ is defined either on the set $\{0, a\} \cup X$ or on $\{0, a, b\} \cup X$, and its operation $*$ is defined to satisfy condition

$$u * 0 = 0 * u = u * a = a * u = 0$$

for every element u in it.

Table 2 The Cayley table for B

$\star$	0	a	b	x	y	z
0	0	0	0	0	0	0
a	0	0	0	0	0	0
b	0	0	0	0	0	0
x	0	0	0	0	a	b
y	0	0	0	b	0	a
z	0	0	0	a	b	0

Table 3 The Cayley table for C

$\star$	0	a	x	y	z
0	0	0	0	0	0
a	0	0	0	0	0
x	0	0	a	0	a
y	0	0	a	a	0
z	0	0	0	a	a

In particular, let $D(n) = \{0, a\} \cup X$, and, for every $x, y \in X$, define

$$x * y = \begin{cases} 0 & \text{if } x = y; \\ a & \text{if } x \neq y. \end{cases}$$

Let $F(n) = \{0, a\} \cup X$ where $(X, \leq)$ is a chain, and, for every $x, y \in X$, put

$$x * y = \begin{cases} 0 & \text{if } x \leq y; \\ a & \text{if } x > y. \end{cases}$$

Let $G(n) = \{0, a, b\} \cup X$ where $(X, \leq)$ is a chain. Define $*$ such that, for all $u \in G(n)$, let

$$u * b = b * u = 0,$$

and for every $x, y \in X$, let

$$x * y = \begin{cases} 0 & \text{if } x = y; \\ a & \text{if } x < y; \\ b & \text{if } x > y. \end{cases}$$

Let $H(n) = \{0, a\} \cup X$. For every $x, y \in X$ we put

$$x * y = a.$$

Let $K(n) = \{0, a\} \cup X$, and, for every $x, y \in X$ we put

$$x * y = \begin{cases} a & \text{if } x = y; \\ 0 & \text{if } x \neq y. \end{cases}$$

Let $R(n) = \{0, a\} \cup X$ where $(X, \leq)$ is a chain, and, for every $x, y \in X$, we put

$$x * y = \begin{cases} 0 & \text{if } x < y; \\ a & \text{if } x \geq y. \end{cases}$$

Let $T(n) = \{0, a, b\} \cup X$. Define $*$ such that, for every $x, y \in X$, let

$$x * y = b; \quad x * b = b * x = a; \quad \{0, a, b\} * \{0, a, b\} = \{0\}.$$

Note that all the above definitions but those of C and $R(n)$ appeared in [2, 3]. Notice that $D(1)$, $F(1)$ and $G(1)$ are null semigroups, and $H(1) = K(1) = R(1)$. Since we intend to list pairwise disjoint classes of semigroups in our main theorem (see below), we require $n \geq 2$ for $D(n)$, $F(n)$, $G(n)$ and $K(n)$, $R(n)$.

Now we formulate the main theorem of the paper.

Theorem 1 *The complete list (up to isomorphism) of finite structurally uniform nilsemigroups is as follows:*

- (1) *a null semigroup;*
- (2) *the nilsemigroup A ;*
- (3) *the nilsemigroup B ;*
- (4) *the nilsemigroup C ;*
- (5) *the nilsemigroup $D(n)$ ($n \geq 2$);*
- (6) *the nilsemigroup $F(n)$ ($n \geq 2$);*
- (7) *the nilsemigroup $G(n)$ ($n \geq 2$);*
- (8) *the nilsemigroup $H(n)$;*
- (9) *the nilsemigroup $K(n)$ ($n \geq 2$);*
- (10) *the nilsemigroup $R(n)$ ($n \geq 2$);*
- (11) *the nilsemigroup $T(n)$.*

3 Proof of the main theorem

Let S denote an arbitrary finite nilsemigroup. For any $x \in S$, the smallest natural number k such that $x^k = 0$ is called the nilpotency index of x and is denoted $\text{nd}(x)$. It is well known that each $\mathcal{J}(= \mathcal{D})$ -class of S is a singleton, and so the preorder $\leq_{\mathcal{J}}$ is a partial order on S with 0 being the least element, and every non-zero principal factor of S is a two-element null semigroup. Immediate consequences are formulated in Proposition 1. For brevity, we denote the poset $(S; \leq_{\mathcal{J}})$ by S' , and the relation $\leq_{\mathcal{J}} \cap \neq$ by $<_{\mathcal{J}}$. The covering relation in S' is denoted by $\prec_{\mathcal{J}}$. As usual, $J(s)$ denotes the principal ideal generated by $s \in S$.

- Proposition 1** (i) *The ideals (the principal ideals) of S are just the order ideals (the principal order ideals) of the poset S' .*
 (ii) *For every positive integer n ($n \leq |S|$), S contains an ideal of cardinality n .*
 (iii) *If $s, u, v \in S$ such that $s = uv$ then either $s <_{\mathcal{J}} u$ and $s <_{\mathcal{J}} v$, or $s = 0$ and at least one of u and v is equal to 0 .*

By Proposition 1(i) and (iii), one can easily deduce the following.

- Proposition 2** (i) *If A is the set of all atoms of S' then the ideal $A^0 = \{0\} \cup A$ is a null semigroup.*
 (ii) *If I is an ideal of S and $u \in S$ such that $\text{nd}(u) = 2$ then $\{u\} \cup I$ is a subsemigroup of S . In particular, $\{u\} \cup A^0$ is a null semigroup.*

The Proposition 1(ii) implies $m(U) = |U|$ for every subsemigroup U in S whence Proposition 3 is straightforward.

Proposition 3 [2, Lemma 5] *A finite nilsemigroup is structurally uniform if and only if its subsemigroups of the same cardinality are isomorphic.*

Denote by ND_i ($i \geq 2$) the class of all structurally uniform nilsemigroups S in which the maximum of the nilpotency index of the elements is i , and if $S \in \text{ND}_i$, let $X = \{x \in S : \text{nd}(x) = i\}$. According to [3, Lemma 5] we have $i \leq 4$. Each member of ND_4 has a subsemigroup generated by an element of nilpotency index 4, and such a subsemigroup contains a three-element null semigroup as a subsemigroup. Also one can easily see that no member S of ND_4 contains an element of nilpotency index 3.

By [2, Proposition 3], the members of ND_2 are (up to isomorphism) just those listed in Theorem 1(1), (2), (3), (5), (6), (7). The commutative members of ND_3 and ND_4 are shown in [3, Theorem 3] to be those in Theorem 1(8),(9) and Theorem 1(11), respectively. In the rest of the proof, we find the non-commutative members of ND_3 , and show that each member of ND_4 is commutative.

Lemma 1 (i) *If $S \in \text{ND}_3$ then there is a unique element $a \in S$ of nilpotency index 2, and $\{0, a\}$ is an ideal of S . We also have $x^2 = a$ and $0 <_{\mathcal{J}} a <_{\mathcal{J}} x$ for any $x \in X$. Equivalently,*

$$S = \{0, a\} \cup X \text{ and } xa = ax = a^2 = 0, \quad x^2 = a \text{ for every } x \in X. \quad (1)$$

(ii) *If $|S| = 4$ and $X = \{x, y\}$ then*

$$xy, yx \in \{0, a\}. \quad (2)$$

Consequently, for every $S \in \text{ND}_3$, (2) holds in each four-element subsemigroup of S for its elements x, y of nilpotency index 3.

Proof (i) The three-element subsemigroups of S are cyclic, therefore we immediately see by Proposition 2 that there is a unique atom a in S' , and it is the only element of S with nilpotency index 2. Due to Proposition 1(iii), the equality $x^2 = a$ ($x \in X$) implies $a <_{\mathcal{J}} x$.

(ii) Straightforward by Proposition 1(iii). $\square$

Lemma 2 *Let $S \in \text{ND}_4$.*

(i) *There is a unique atom a and a unique cover b of a in S' , and $a, b \in S$ are the only elements of nilpotency index 2. Both $\{0, a\}$ and $\{0, a, b\}$ are ideals of S and are null semigroups. Moreover, we have $x^2 = b$, $x^3 = a$ and $0 <_{\mathcal{J}} a <_{\mathcal{J}} b <_{\mathcal{J}} x$ for any $x \in X$. Equivalently,*

$$S = \{0, a, b\} \cup X, \quad xa = ax = ba = ab = a^2 = b^2 = 0$$

$$\text{and } x^2 = b, \quad xb = bx = a \text{ for every } x \in X. \quad (3)$$

(ii) *If $|S| = 5$ and $X = \{x, y\}$ then*

$$xy = yx = b. \quad (4)$$

Consequently, for every $S \in \text{ND}_4$, (4) holds in each five-element subsemigroup of S for its elements x, y of nilpotency index 4.

Proof (i) By Proposition 2(i) there are at most two atoms in S' . If there are two then denote them a and b . If there is only one atom a then choose and fix a cover b of a . In both cases $J(a) \cup J(b) = \{0, a, b\}$ is an ideal in S , and this implies by Proposition 2(ii) that if $u \in S - \{0, a\}$ with $\text{nd}(u) = 2$ then $\{0, a, b, u\}$ forms a four-element subsemigroup in S . Since the four-element subsemigroups in S are cyclic but this semigroup is not, we obtain that the only elements of S with nilpotency index 2 are a and b , and consequently, $S - \{0, a, b\} = X$ and $\{x^2, x^3\} = \{a, b\}$ for every $x \in X$. Since a is an atom, we have $xa = 0$, and therefore $a \neq x^2$. Thus $a = x^3$ and $b = x^2$ follow for any $x \in X$. Hence $xb = bx = a$ and this implies that a is the only atom in S' and $0 <_{\mathcal{J}} a <_{\mathcal{J}} b$. Finally, the equation $x^2 = b$ ($x \in X$) implies $b <_{\mathcal{J}} x$ by Proposition 1(iii).

(ii) It is obvious by Proposition 1(iii) that $xy, yx \in \{0, a, b\}$, and it follows from (i) that $x(xy) = x^2y = by = a$. Since the only element $c \in \{0, a, b\}$ such that $xc = a$ is $c = b$, we deduce that $xy = b$. Similarly, $yx = b$ also holds. $\square$

Now let $S \in \text{ND}_i$ ($i \in \{3, 4\}$). Lemmas 1(i) and 2(i) show that $M = S - X$ is a principal ideal. For uniformity, we denote its generating element by m (i.e., if $S \in \text{ND}_3$ then $m = a$ and if $S \in \text{ND}_4$ then $m = b$). Recall that $m <_{\mathcal{J}} x$ for every $x \in X$. Notice that Lemmas 1(i) and 2(i) uniquely determine the products st in S where either s or t belongs to M , or $s = t \in X$. For our later convenience, we denote the missing part of the Cayley table, that is, the part containing the products st where $s, t \in X$ and $s \neq t$ by $\mathcal{X}_S$. Now we turn to investigating these products.

Lemma 3 *Let $S \in \text{ND}_i$ ($i \in \{3, 4\}$).*

(i) *We have $m <_{\mathcal{J}} x$ in S' for every $x \in X$, and*

$$xy \in M \text{ for any } x, y \in X \text{ with } x \neq y \quad (5)$$

- (ii) The subsemigroups of S with cardinality at least $i - 1$ are just the subsets $M \cup U$ where $U \subseteq X$.

Proof (i) If there is an element $z \in X$ not covering m in S' then there exist $x, y \in X$ such that $m \prec_{\mathcal{J}} x \prec_{\mathcal{J}} y$. Then $J(y) = M \cup \{x, y\}$ is an $(i + 1)$ -element subsemigroup of S whence it follows by Lemmas 1(ii) and 2(ii) that $xy, yx \in M$. Since $x \in J(y)$, we have $x = uyv$ for some $u, v \in S^1$ where at least one of u and v belongs to S . If $u \in S$ then $x \prec_{\mathcal{J}} uy \prec_{\mathcal{J}} u, y$ are implied by Proposition 1(iii) whence $u \in X$ and $x = uy$ follow since $x \prec_{\mathcal{J}} y$. Similarly, if $v \in S$ then $v \in X$ and $x = yv$. Suppose that $x = uy$, the other possibility is similar. By applying Lemmas 1(ii) and 2(ii), we obtain that $m = x^2 = (uy)x = u(yx) \in uM$, and Lemmas 1(i) and 2(i) show that $uM \subseteq M - \{m\}$. This contradiction shows that all elements of X cover m in S' . Consequently, for every $x \in X$, we have $J(x) = M \cup \{x\}$, the subsemigroup generated by x , and for every $x, y \in X$ with $x \neq y$, the ideal $J(x) \cup J(y) = M \cup \{x, y\}$ is an $(i + 1)$ -element subsemigroup of S . Thus Lemmas 1(ii) and 2(ii) imply (5). □

(ii) Straightforward by (i). □

Lemma 4 Every member of ND_4 is commutative.

Proof Let $S \in \text{ND}_4$. By (3) we have $su = us$ for any $s \in S$ and $u \in M$. Lemma 3(ii) implies that, for any $x, y \in X$ with $x \neq y$, $M \cup \{x, y\}$ is a five-element subsemigroup in S , and Lemma 2(ii) implies that $xy = yx = b$. □

Remark (i) All subsemigroups of any $S \in \text{ND}_i$ ($i \in \{3, 4\}$) are ideals in S . For, each cyclic subsemigroup of any such S is a principal ideal in S by Lemmas 1–3.
 (ii) Although Theorem 1(11) is straightforward by Lemma 4 and [3, Theorem 3], it is worth mentioning that the proof of Lemma 4 can be easily extended to verifying that each member S of ND_4 is isomorphic to $T(|S| - 3)$. For, the equality $xy = yx = b$ combined with (3) shows that S is an inflation of any of the subsemigroups $J(x)$ ($x \in X$) where the inflating set corresponding to x is X ($|X| = |S| - 3$), and the rest of the inflating sets are singletons. All such inflations of the same cardinality are obviously isomorphic and structurally uniform, and the semigroups $T(n)$ ($n \geq 1$) ($|T(n)| = n + 3$) are among them by definition.

Now we turn to completing the proof for the class ND_3 . Let S be an arbitrary non-commutative member of ND_3 . By Lemmas 1(i) and 3(i) properties (1) and (5) hold, and $|S| \geq 4$, $|X| = |S| - 2 \geq 2$. For our later convenience, we put $n = |X|$ and assume that $X = \{x_1, \dots, x_n\}$.

Notice that any Cayley table satisfying conditions (1) and (5) defines an associative operation since all three-factor products are equal to 0. Moreover, condition (1) uniquely determines the Cayley table of S up to $\mathcal{X}_S$, and by (5) the entries of $\mathcal{X}_S$ are 0 and a .

Clearly, S is commutative if and only if $\mathcal{X}_S$ is symmetric. Therefore finding all non-commutative members of ND_3 up to isomorphism amounts to filling in $\mathcal{X}_S$ in a non-symmetric way, so that all essentially different Cayley tables of non-commutative members of ND_3 are obtained.

We make a stop to consider the family $R(n)$ ($n \geq 2$).

Lemma 5 For any n ($n \geq 2$), $R(n)$ is a non-commutative member of ND_3 , and its $(n - 1)$ -element subsemigroups are isomorphic to $R(n - 1)$.

Proof It is straightforward from the definition that $R(n)$ is non-commutative and possesses properties (1) and (5), and consequently, also that formulated in Lemma 3(ii). Hence $R(n)$ is a semigroup and all the properties to be proved but structural uniformity follow. Taking into account also Lemma 1(ii), it is also clear that the subsemigroups of cardinality k with $k \leq 4$ are isomorphic. In particular, $R(2)$ is structurally uniform. It is also easy to see by definition and Lemma 3(ii) that if $n \geq 3$ then each subsemigroup of cardinality $n - 1$ is isomorphic to $R(n - 1)$ so that a routine induction completes the proof. $\square$

For our later convenience notice that if $n \geq 2$ and an $(n + 2)$ -element semigroup S satisfying (1) and (5) is isomorphic to $R(n)$ then $\mathcal{X}_S$ has the following properties: it is 'antisymmetric' in the sense that, for every pair of symmetric entries, one is 0 and the other is a , and for every j ($0 \leq j \leq n - 1$), there is a (unique) column in $\mathcal{X}_S$ where the number of the 0's is j .

Returning to an arbitrary non-commutative $S \in \text{ND}_3$, it is straightforward from the observations on their Cayley tables that if $n = 2$ then S is isomorphic to $S = R(2)$. Moreover, for any non-commutative $S \in \text{ND}_3$, the four-element subsemigroups of S are isomorphic to $R(2)$.

In particular, if $|S| = 5$ then $S = R(2) \cup \{x_3\}$. Checking the possibilities how one can fill in the entries of $\mathcal{X}_S$ such that both $M \cup \{x_1, x_3\}$ and $M \cup \{x_2, x_3\}$ be isomorphic to $R(2)$ one obtains that $S = C$ or $S = R(3)$.

We proceed by considering the semigroups $S \in \text{ND}_3$ having C and having $R(3)$ as a subsemigroup separately. Similarly to the observation on the Cayley tables of the semigroups isomorphic to $R(n)$ ($n \geq 2$), let us notice that a five-element semigroup S satisfying (1) and (5) is isomorphic to C if and only if in each row and in each column of $\mathcal{X}_S$ one of the two entries is 0 and the other is a . Hence $\mathcal{X}_S$ is uniquely determined by any of its entries.

Lemma 6 No semigroup $S \in \text{ND}_3$ contains a proper subsemigroup isomorphic to C .

Proof It suffices to verify that there is no six-element $S \in \text{ND}_3$ with C as a subsemigroup. Let $S = M \cup \{x_1, x_2, x_3, x_4\}$ be a semigroup satisfying (1) and (5), and assume that $M \cup \{x_1, x_2, x_3\} = C$. The rest of the five-element subsemigroups of S are $T_{ij} = M \cup \{x_i, x_j, x_4\}$ where $1 \leq i < j < 4$. Suppose that T_{12} is isomorphic to C . Filling in the entries of $\mathcal{X}_{T_{12}}$ by applying that $x_1x_2 = 0$ we obtain that $x_2x_4 = a = x_2x_3$. Thus T_{23} is not isomorphic to C , and so $S \notin \text{ND}_3$. $\square$

Lemma 7 Let $S \in \text{ND}_3$ be such that $|S| \geq 5$ and S contains a subsemigroup isomorphic to $R(3)$. Then S is isomorphic to $R(|S| - 2)$.

Proof The statement is obvious if $|S| = 5$, that is, if $n = 3$. We proceed by induction on n . Suppose that, for some n ($n \geq 3$), each $(n + 2)$ -element member of ND_3 containing $R(3)$ as a subsemigroup is isomorphic to $R(n)$, and let $S \in \text{ND}_3$ such that $|S| = n + 3$ and it contains a subsemigroup isomorphic to $R(3)$. By the induction hypothesis, we can assume that S has $R(n)$ as a subsemigroup. Thus $S = R(n) \cup \{x_{n+1}\}$ and

the missing entries in the Cayley table of S are those in the last row and the last column of $\mathcal{X}_S$. Since S is structurally uniform, the subsemigroups $M \cup (X - \{x_n\})$ and $M \cup (X - \{x_1\})$ are isomorphic to $R(n)$. Applying the observation on the Cayley tables of such semigroups, we obtain from the first one that $x_i x_{n+1} = x_i x_n = 0$ and $x_{n+1} x_i = x_n x_i = a$ for $i = 1, 2, \dots, n-1$, and from the second that $x_n x_{n+1} = x_{n-1} x_n = 0$ and $x_{n+1} x_n = x_n x_{n-1} = a$. Thus $S = R(n)$. $\square$

It is easy to verify the validity of the following combinatorial equalities:
 $|\text{Sub}(R(n))| = 1 + 2^n$, $|\text{LAut}(R(n))| = \binom{2n}{n} + 1$, $|\text{Sub}(T(n))| = 3 + 2^n$,
 $|\text{LAut}(T(n))| = \sum_{k=0}^n \binom{n}{k}^2 \cdot k! + 6$.

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