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HYPERINERTIA AND SUBPOTENTIALITY IN GENERALIZED LAGRANGIAN AND LAGRANGIAN-MAXWELL EQUATIONS FOR DISSIPATIVE DYNAMIC SYSTEMS CONSIDERING THE INTERACTIVITY OF RAYLEIGH DISSIPATION FUNCTIONS

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Abstract

Based on the deepening of the fundamental concepts of force and energy, this paper presents a generalization of the Lagrangian and Lagrangian-Maxwell equations for dissipative dynamic systems of mechanical and electromagnetic origin. In the mathematical aspect, their complete form and internal structure are obtained and presented for the first time, while in the physical aspect, the concepts of hyperinertia, subpotentiality, and interactive energy dissipation are introduced and partially disclosed. The obtained results complement the initial framework of the theory of dynamic systems.

Keywords: generalized Lagrangian and Lagrangian-Maxwell equations, dissipative dynamic systems, theoretical fundamentals of electrical engineering, theoretical mechanics, hyperinertial and subpotential forces and energies, generalized dissipative forces and Rayleigh functions

Анотація

В роботі на основі поглиблення фундаментальних понять сили та енергії проведено узагальнення рівнянь Лагранжа та Лагранжа-Максвелла для дисипативних динамічних систем механічного та електромагнітного походження: в математичному аспекті вперше отримано та представлено їх завершену форму та внутрішню структуру, а у фізичному введено та частково розкрито поняття надінерційності, субпотенціальності, інтерактивної дисипації енергії. Отримані результати доповнюють вихідний базис теорії динамічних систем.

Ключові слова: узагальнені рівняння Лагранжа та Лагранжа-Максвелла, дисипативні динамічні системи, теоретичні основи електротехніки, теоретична механіка, надінерційні та субпотенціальні сили і енергії, інтерактивні дисипативні сили та функції Релея

Let us consider dissipative dynamic systems with an arbitrary number of degrees of freedom and focus our attention on such fundamental properties as the reversibility and irreversibility of energy interconversions observed in these systems during their motion (state transition) under the action of external generalized forces.

As an example, we shall consider the motion of dynamic systems of mechanical and electromagnetic nature, in both cases being lumped-parameter systems.

It is known that the motion of mechanical systems with an arbitrary number of degrees of freedom n can be described by the Lagrangian equations [1, 2]:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s}+\frac{\partial \Phi}{\partial \dot{q}_s}+\frac{\partial \Pi}{\partial q_s}=Q_s, \quad s=1,2,...,n, \tag{1}$$

while the motion of dynamic systems of electromagnetic origin, in particular, electrical circuits of an arbitrary order, can be described by the Lagrangian-Maxwell equations [3]:

$$\frac{d}{dt}\frac{\partial W_м}{\partial \dot{q}_s}+\frac{\partial \Phi}{\partial \dot{q}_s}+\frac{\partial W_e}{\partial q_s}=e_s, \quad s=1,2,...,n. \tag{2}$$

In the aforementioned systems, along with generalized potential $\left(\frac{\partial \Pi}{\partial q_s}\text{ or }\frac{\partial W_e}{\partial q_s}\right)$ and inertial $\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s}\text{ or }\frac{d}{dt}\frac{\partial W_м}{\partial \dot{q}_s}\right)$ forces, forces of a dissipative nature $\left(\frac{\partial \Phi}{\partial \dot{q}_s}\right)$ are simultaneously present.

As can be seen from the Lagrangian and Lagrangian-Maxwell equations, these forces are defined through the eponymous Rayleigh dissipation function (otherwise known as the scattering function) [2].

By its physical meaning, the time integral of the Rayleigh dissipation function determines half of the energy dissipated in the dynamic system; hence, the Rayleigh dissipation function itself, as a derivative, manifests as half of the power of the dissipated energy.

For linear or linearized mechanical systems with an arbitrary number of degrees of freedom *n*, the Rayleigh dissipation function can be written as:

$$\Phi = \frac{1}{2}\sum_{s=1}^{n}\sum_{p=1}^{n}(\pm)\alpha_{sp}\dot{q}_s\dot{q}_p, \tag{3}$$

and for electrical circuits of an arbitrary (*2n*) order as:

$$\Phi = \frac{1}{2}\sum_{s=1}^{n}\sum_{p=1}^{n}(\pm)R_{sp}\dot{q}_s\dot{q}_p \tag{4}$$

respectively [4–8].

Relations (3) and (4) provide a deeper insight into the mechanism of total energy dissipation in a system during its motion. It directly follows from these relations that energy dissipation is observed in those dynamic systems which, due to characteristic physical phenomena, exhibit a capacity to exert active force opposition to any change in a generalized coordinate depending on the rate of such change (generalized velocity), but independently of the rate of change of the said velocity itself (generalized acceleration).

The latter implies that a limited range of dissipative physical phenomena and processes is taken into account when determining the Rayleigh function. For instance, among the well-known examples, one can distinguish various manifestations of friction in mechanical systems or the electrical resistance of resistive elements in electrical circuits.

Now, let us imagine that a dynamic system lacked or was entirely unaffected by all (!) physical phenomena of a dissipative nature, even those mentioned above and determined by the generalized velocity. In such a case, the phenomenon of energy dissipation itself in the dynamic system would be unaccounted for and, presumably, unknown.

The classical Lagrangian (1) and Lagrangian-Maxwell (2) equations prove precisely this. However, they also prompt a question: do we currently know enough about the complex course of physical phenomena characterized by energy dissipation? The answer is unambiguous. No! Naturally, we do not know everything...

And if so, the next question logically arises. The pivotal one. Are all dissipative forces, by their structure, accounted for in the classical Lagrangian and Lagrangian-Maxwell equations themselves?

The author of these abstracts does not dare to state categorically, 'No.'

However, neither can they say 'Yes,' as they fully admit that in dissipative dynamic systems of various physical origins (including, in particular, those mentioned above), as-yet-unexplored or unknown physical phenomena may be observed. In such phenomena, the active resistance of the dynamic system would manifest itself not only against the change in the generalized coordinate (at its given velocity) but also against the change in this velocity itself (at its given generalized acceleration), as well as against the change in their higher-order time derivatives. Possibly, even against the change in the integrals of the generalized coordinate. Dissipative forces arising from the resistance to changes in such motion parameters of the system will, naturally, possess a complex, internally active (i.e., interactive) structural nature.

All the aforementioned implies that higher-order dissipative forces are neither accounted for nor described by the classical Lagrangian and Lagrangian-Maxwell equations in terms of form, structure, or content.

However, the author is fully aware that from a physical standpoint, without conclusive (theoretical or experimental) confirmation of the existence—or at least the possibility of existence—of the specified physical phenomena, there are no grounds, no justification, and no reason to claim the limited generality or incompleteness of the Lagrangian and Lagrangian-Maxwell equations. Nevertheless...

However, from a mathematical perspective, one cannot fail to see that the logical rigor of the classical Lagrangian and Lagrangian-Maxwell equations is not exhaustive and requires generalization!

And if we take the deepening of the foundational concepts of force and energy (including its dissipation) as a basis, such a generalization becomes both possible and permissible [9, 10].

In particular, for the dissipative dynamic systems under consideration, the mathematical form of the generalized Lagrangian and generalized Lagrangian-Maxwell equations can and should be a system of integro-differential equations of the following form:

$$\begin{gathered}\left(\cdots+\frac{d^3}{dt^3}\frac{\partial W_3}{\partial q_s^{(3)}}+\frac{d^2}{dt^2}\frac{\partial \Phi_2}{\partial q_s^{(3)}}+\frac{d^2}{dt^2}\frac{\partial W_2}{\partial \ddot{q}_s}+\frac{d}{dt}\frac{\partial \Phi_1}{\partial \ddot{q}_s}\right)+\\ +\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s}+\frac{\partial \Phi_0}{\partial \dot{q}_s}+\frac{\partial \Pi}{\partial q_s}\right)+\\ +\left(\int\frac{\partial \Phi_{-1}}{\partial q_s}dt+\int\frac{\partial W_{-1}}{\partial\left(\int q_s\,dt\right)}dt+\int\left(\int\frac{\partial\,\Phi_{-2}}{\partial\left(\int q_s\,dt\right)}dt\right)dt+\int\left(\int\frac{\partial W_{-2}}{\partial\left(\int\left(\int q_s\,dt\right)dt\right)}dt\right)dt+\cdots\right)=\Theta_s,\\ s=1,2,...,n,\end{gathered}\tag{5}$$

which, upon introducing the differentiation $D^m=\frac{d^m}{dt^m}$, $m=0,1,\ldots$ and integration $D^m=\int\cdots\left(\int\overset{m}{\cdots}dt\right)\cdots dt$, $m=-1,-2,\ldots$ operators into it, becomes more compact in its formal representation and can be written as:

$$\begin{gathered}\left(\cdots+D^3\frac{\partial W_3}{\partial D^3 q_s}+D^2\frac{\partial \Phi_2}{\partial D^3 q_s}+D^2\frac{\partial W_2}{\partial D^2 q_s}+D^1\frac{\partial \Phi_1}{\partial D^2 q_s}\right)+\\ +\left(D^1\frac{\partial T}{\partial D^1 q_s}+D^0\frac{\partial\,\Phi_0}{\partial D^1 q_s}+D^0\frac{\partial \Pi}{\partial D^0 q_s}\right)+\\ +\left(D^{-1}\frac{\partial \Phi_{-1}}{\partial D^0 q_s}+D^{-1}\frac{\partial W_{-1}}{\partial D^{-1} q_s}+D^{-2}\frac{\partial\,\Phi_{-2}}{\partial D^{-1} q_s}+D^{-2}\frac{\partial W_{-2}}{\partial D^{-2} q_s}+\cdots\right)=\Theta_s,\\ s=1,2,...,n,\end{gathered}\tag{6}$$

or

$$\begin{gathered}\sum_{m=-\infty}^{+\infty}D^m\left(\frac{\partial \Phi_m}{\partial D^{m+1}q_s}+\frac{\partial W_m}{\partial D^m q_s}\right)=\Theta_s,\\ s=1,2,...,n.\end{gathered}\tag{7}$$

Differential equations (5)–(7) represent a generalized form with respect to the classical Lagrangian (1) and Lagrangian-Maxwell (2) equations. Therefore, the solutions to the *fundamental Cauchy problem* will undoubtedly possess deeper and, possibly, as-yet-unknown *mathematical* meanings. This is true if only because the *generalized Lagrangian and Lagrangian-Maxwell* equations (5)–(7) incorporate not only the known *potential* $\left(V^m=D^m\frac{\partial W_m}{\partial D^m q_s},\ m=0\right)$, *inertial* $\left(V^m=D^m\frac{\partial W_m}{\partial D^m q_s},\ m=1\right)$, and *dissipative* $\left(\frac{\partial \Phi_m}{\partial D^{m+1}q_s},\ m=0\right)$ *forces*, *energies* $\left(W_m,\ m=0,1\right)$, and the *Rayleigh function* $\left(\Phi_m,\ m=0\right)$ defined by the classical equations, but also the hitherto unknown *hyperinertial forces*:

$$\begin{gathered}V^m=D^m\frac{\partial W_m}{\partial D^m q_s}=\frac{d^m}{dt^m}\frac{\partial W_m}{\partial q_s^{(m)}},\ m=2,3,\ldots,\\ s=1,2,...,n\end{gathered}\tag{8}$$

and their associated *energies* $\left(W_m,\ m=2,3,\ldots\right)$, *subpotential forces*:

$$V^m = D^m \frac{\partial W_m}{\partial D^m q_s} = \int \left(\overset{m}{\cdots} \int \frac{\partial W_m}{\partial \left(\int \overset{m}{\cdots} \left(\int q_s \, dt \right) dt \right)} d\,t \right) d\,t, \, m = -1, -2, \ldots, \quad (9)$$

$$s = 1,2,...,n$$

and energies $\left(W_m, m = -1, -2, \ldots\right)$, as well as *hyperinertial* $\left(\Phi_m, m = 1, 2, \ldots\right)$ and *subpotential* $\left(\Phi_m, m = -1, -2, \ldots\right)$ *Rayleigh dissipation functions*, and *interactive dissipative forces* determined through these Rayleigh functions:

$$V^m = D^m \frac{\partial \Phi_m}{\partial D^{m+1} q_s} = \frac{d^m}{d\,t^m} \frac{\partial \Phi_m}{\partial q_s^{(m+1)}}, \, m = 1, 2, \ldots, \quad (10)$$

$$s = 1,2,...,n$$

and

$$V^m = D^m \frac{\partial \Phi_m}{\partial D^{m+1} q_s} = \int \left(\overset{m}{\cdots} \int \frac{\partial \Phi_m}{\partial \left(\int \overset{m+1}{\cdots} \left(\int q_s \, dt \right) dt \right)} d\,t \right) d\,t, \, m = -1, -2, \ldots, \quad (11)$$

$$s = 1,2,...,n$$

respectively, which remain undefined by the classical Lagrangian and Lagrangian-Maxwell equations.

Conclusions

Based on the deepening of the fundamental concepts of force and energy, this paper presents a generalization of the Lagrangian and Lagrangian-Maxwell equations for dissipative dynamic systems of mechanical and electromagnetic origin. In the mathematical aspect, their complete form and internal structure are obtained and presented for the first time, while in the physical aspect, the concepts of hyperinertia, subpotentiality, and interactive energy dissipation are introduced and partially disclosed.

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