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GENERALIZED LAGRANGE AND LAGRANGE-MAXWELL EQUATIONS FOR CONSERVATIVE DYNAMICAL SYSTEMS CONSIDERING THE ACTION OF OVER-INERTIAL AND SUBPOTENTIAL FORCES

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Abstract

Based on a deeper insight into the fundamental concepts of force and energy, this paper generalizes the Lagrange and Lagrange-Maxwell equations for conservative dynamical systems of mechanical and electromagnetic origin. In the mathematical aspect, their complete form and internal structure are obtained and presented for the first time. In the physical aspect, the concepts of over-inertial and subpotential behavior are introduced and partially disclosed. The obtained results complement the original baseline of dynamical systems theory.

Keywords: generalized Lagrange and Lagrange-Maxwell equations, conservative dynamical systems, theoretical fundamentals of electrical engineering, theoretical mechanics, over-inertial and subpotential forces and energies, over-inertial and subpotential behavior.

Анотація

В роботі на основі поглиблення фундаментальних понять сили та енергії проведено узагальнення рівнянь Лагранжа та Лагранжа-Максвелла для консервативних динамічних систем механічного та електромагнітного походження: в математичному аспекті вперше отримано та представлено їх завершену форму та внутрішню структуру, у фізичному аспекті введено та частково розкрито поняття надінерційності та субпотенціальності. Отримані результати доповнюють вихідний базис теорії динамічних систем.

Ключові слова: узагальнені рівняння Лагранжа та Лагранжа-Максвелла, консервативні динамічні системи, теоретичні основи електротехніки, теоретична механіка, надінерційні та субпотенціальні сили і енергії, надінерційність та субпотенціальність

In 1788, the prominent mathematician and mechanician Joseph-Louis Lagrange, in his seminal work *Mécanique Analytique* [1], presented the fundamental differential equations of classical mechanics. These equations describe the motion of a mechanical system with an arbitrary degree of freedom through the change in its total energy, defined relative to a given set of generalized coordinates.

For conservative dynamical systems, the Lagrange equations can be written in the form:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s}+\frac{\partial \Pi}{\partial q_s}=0, \quad s=1,2,...,n, \tag{1}$$

where q_s is the s-th generalized coordinate, T and Π are the kinetic and potential energies of the mechanical system, respectively, and n is the number of its degrees of freedom.

About a century later, in 1873, James Clerk Maxwell perceived an affinity in the state-changing dynamics of electromagnetic systems. Having identified profound mathematical and physical analogies between electromagnetism and mechanics, in his treatise *A Treatise on Electricity and Magnetism* [2], alongside the celebrated equations describing the fundamental properties of the electromagnetic field (Maxwell's equations), he derived the differential equations of motion for dynamical systems of electromechanical and electromagnetic nature. In terms of both external and internal structure, these equations proved to be analogs of the Lagrange equations.

In particular, for electrical circuits (which are inherently also dynamical systems, albeit of electromagnetic origin), under the precondition of no electromagnetic energy dissipation (it should be recalled that strictly conservative systems are considered herein), and in the absence of external generalized forces (in this context, energy sources), the Lagrange-Maxwell equations in the "force-voltage" generalized coordinate system (the first system of electro-dynamic analogies) take the form:

$$\frac{d}{dt}\frac{\partial W_{м}}{\partial \dot{q}_s}+\frac{\partial W_e}{\partial q_s}=0, \quad s=1,2,...,n, \tag{2}$$

where q_s is the time integral of the mesh (loop) current of the s independent loop of the electrical circuit (i.e., the mesh charge acting as the s-th generalized coordinate), $W_{м}$ and W_e are the energies of the magnetic and electric fields of the circuit under consideration, respectively, and n is its number of degrees of freedom.

The author deems it necessary to note that both the Lagrange equations (1) and the Lagrange-Maxwell equations (2), firstly, with respect to conservative systems, reveal the action of only two physical types of forces—forces of a potential nature and inertial forces—and, secondly, establish a physical parity between them: neither of these forces can be represented as a manifestation of the other. At the same time, in both the Lagrange and Lagrange-Maxwell equations, these generalized forces are mathematically composite and are determined in each case through two fundamental but distinct types of energy, which ultimately manifests as the aforementioned diversification of the forces themselves.

Without entering into the proxy debate between Isaac Newton [3] and Joseph-Louis Lagrange [1] regarding priority between acting forces and the available energies associated with them, one cannot fail to notice that the emergence of forces in conservative systems manifests itself, firstly, exclusively upon a change in the energies specified in equations (1) and (2), and, secondly, in different ways. In the case of potential forces ($\frac{\partial \Pi}{\partial q_s}$ or $\frac{\partial W_e}{\partial q_s}$), this occurs upon a change in the generalized coordinates q_s, which may or may not cause (depending on the properties (!) of the system) a change in the potential energy — for instance, for a linear mechanical system:

$$\Pi=\frac{1}{2}\sum_{s=1}^{n}\sum_{p=1}^{n}(\pm)\beta_{sp}q_sq_p, \tag{3}$$

or the electric field energy in a linear lumped-parameter electrical circuit:

$$W_e=\frac{1}{2}\sum_{s=1}^{n}\sum_{p=1}^{n}(\pm)C_{sp}^{-1}q_sq_p. \tag{4}$$

Conversely, in the case of inertial forces, their emergence occurs upon a change in the generalized velocities (the first time derivatives of the generalized coordinates) $\dot{q}_s$, which, again, may or may not cause (likewise depending on the properties (!) of the system) a change in the kinetic energy of a mechanical system:

$$T=\frac{1}{2}\sum_{s=1}^{n}\sum_{p=1}^{n}(\pm)m_{sp}\dot{q}_s\dot{q}_p, \tag{5}$$

or the magnetic field energy of an electrical circuit:

$$W_{м}=\frac{1}{2}\sum_{s=1}^{n}\sum_{p=1}^{n}(\pm)L_{sp}\dot{q}_s\dot{q}_p, \tag{6}$$

where the rate of these changes over time serves as an additional accompanying factor [4-7]. If the parameters characterizing the aforementioned properties of the system (for example, in relations (3)–(6)) are equal to zero, both the corresponding energy and the force will be absent.

Therefore, the most critical aspect of the aforementioned, in the author's opinion, is the ability (property) of a dynamical system to respond accordingly to a change in at least one of the two interconnected characteristics of its state — namely, a change in position q_s and a change in velocity $\dot{q}_s=\frac{dq_s}{dt}$, typically acting in opposition to such changes.

For instance, if independent dynamical elementary links in mechanical systems lacked the property that classical mechanics defines as mass, the kinetic energy of these links would be absent, regardless of the acceleration (i.e., change in velocity) with which these links moved. As a consequence (or as a cause), inertial forces would also be absent. The system would exhibit entirely non-inertial behavior; thus, the concept of inertia within such a system would remain uninvestigated and, presumably, unknown.

Another example: in the absence of inductive elements with inductances L_{sp}, a similar situation would be observed in electrical circuits with the topological structure of the first system of generalized electro-

dynamic coordinates [4, 8]. During motion within the state space of such dynamical systems, due to the lack of magnetic field energies and their associated voltages (inertial forces), the concept of inertial processes in an electrical circuit would be, at the very least, obscure. This is because everything available for observation in such systems would be limited strictly to the electric fields of capacitive elements and, accordingly, their energies and the voltages (potential generalized forces) associated with these energies.

Therefore, taking the above into account, the author takes the liberty to assert that mathematically and physically, the Lagrange equations and the Lagrange-Maxwell equations are incomplete in form, structure, and content alike. The logical power of these equations is not absolute and requires generalization.

In the mathematical aspect, such a generalization is indeed possible [9, 10].

In particular, for the conservative dynamical systems under consideration, the mathematical form of the generalized Lagrange and generalized Lagrange-Maxwell equations can and should be a system of integro-differential equations of the form:

$$\begin{gathered}\left(\cdots+\frac{d^3}{dt^3}\frac{\partial W_3}{\partial q_s^{(3)}}+\frac{d^2}{dt^2}\frac{\partial W_2}{\partial \ddot{q}_s}\right)+\\ +\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s}+\frac{\partial \Pi}{\partial q_s}\right)+\\ +\left(\int\frac{\partial W_{-1}}{\partial\left(\int q_s\,dt\right)}dt+\int\left(\int\frac{\partial W_{-2}}{\partial\left(\int\left(\int q_s\,dt\right)dt\right)}dt\right)dt+\cdots\right)=0,\\ s=1,2,...,n.\end{gathered} \tag{7}$$

Upon introducing the differentiation operators $D^m=\frac{d^m}{dt^m}$, $m=0,1,\ldots$ and integration operators $D^m=\int\cdots\left(\int\cdots^{m}dt\right)\cdots dt$, $m=-1,-2,\ldots$ this system becomes more compact in its formal representation and can be written as:

$$\begin{gathered}\sum_{m=2}^{+\infty}D^m\frac{\partial W_m}{\partial D^m q_s}+\left(D^1\frac{\partial T}{\partial D^1 q_s}+D^0\frac{\partial \Pi}{\partial D^0 q_s}\right)+\sum_{m=-\infty}^{-1}D^m\frac{\partial W_m}{\partial D^m q_s}=0,\\ s=1,2,...,n,\end{gathered} \tag{8}$$

or

$$\begin{gathered}\sum_{m=-\infty}^{+\infty}D^m\frac{\partial W_m}{\partial D^m q_s}=0,\\ s=1,2,...,n.\end{gathered} \tag{9}$$

From a physical perspective, the situation is substantially more complex since, for conservative dynamical systems, only potential and inertial forces $\left(V^m=D^m\frac{\partial W_m}{\partial D^m q_s},\ m=0,1\right)$ and energies $\left(W_m,\ m=0,1\right)$ are currently known, as reflected in the classical Lagrange (1) and Lagrange-Maxwell (2) equations. Conversely, forces and energies of a different physical nature that comprise the generalized Lagrange and Lagrange-Maxwell equations (7)–(9) naturally remain uninvestigated to date.

These include the potential physical manifestations of, on the one hand, *over-inertial forces*:

$$\begin{gathered}V^m=D^m\frac{\partial W_m}{\partial D^m q_s}=\frac{d^m}{dt^m}\frac{\partial W_m}{\partial q_s^{(m)}},\ m=2,3,\ldots,\\ s=1,2,...,n\end{gathered} \tag{10}$$

and their associated *energies* W_m, $m=2,3,\ldots$, and, on the other hand, *subpotential forces*:

$$V^m = D^m \frac{\partial W_m}{\partial D^m q_s} = \int \left(\overset{m}{\cdots} \int \frac{\partial W_m}{\partial \left(\int \overset{m}{\cdots} \left(\int q_s \, dt \right) dt \right)} dt \right) dt, m = -1, -2, \ldots, \quad (11)$$

$$s = 1,2,...,n$$

and *energies* $W_m, m = -1, -2, \ldots$.

The physical meaning of the components in the generalized Lagrange and Lagrange-Maxwell equations (7)–(9) is that, much like potential and inertial forces (and energies), over-inertial and subpotential forces can emerge—and their corresponding energies can change—whenever variations occur not only in the generalized coordinates or their first derivatives (generalized velocities), but also: firstly (in the case of over-inertial forces (10) and energies), in higher-order time derivatives of the generalized coordinates (for instance, with a change in generalized acceleration); and, secondly (in the case of subpotential forces (11) and energies), in the first and higher-order time integrals of the generalized coordinate, provided that the dynamical system physically possesses such properties.

Therefore, if such over-inertial and subpotential forces and energies do exist in nature, the primary task across various fields of science (specifically including both theoretical mechanics and electromagnetism theory) must be the identification and investigation of the physical phenomena and laws that will ultimately reveal the physical meaning of over-inertial and subpotential behavior within the overarching picture of the world. The generalization of the Lagrange and Lagrange-Maxwell equations serves precisely this purpose.

Conclusions

Based on a deeper insight into the fundamental concepts of force and energy, this work has generalized the Lagrange and Lagrange-Maxwell equations for conservative dynamical systems of mechanical and electromagnetic origin. From a mathematical standpoint, their complete form and internal structure have been obtained and presented for the first time. From a physical perspective, the concepts of over-inertial and subpotential behavior have been introduced and partially disclosed. The results obtained complement the original baseline of dynamical systems theory.

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