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Inverse inference based on interpretable constrained solutions of fuzzy relational equations with extended *max-min* composition

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Abstract. In this paper, we propose a method for solving the System of Fuzzy Relation Equations (SFRE) with extended *max-min* composition for inverse inference problems. The properties of interval and constrained solutions with granular and relational structure of the solution set are investigated. The extended *max-min* SFRE can be represented in the form of the *max-min* subsystems aggregated using the *min* operator or dual *min-max* subsystems aggregated using the *max* operator. When decomposing the SFRE, the set of solutions can be decomposed into the lower and upper subsets bounded by the same aggregating solutions. Each lower (upper) subset is defined by the unique greatest (least) or aggregating solution and the set of minimal (maximal) solutions. Following (Bartl et al. 2012), to avoid excessive granularity and ensure interpretability of the interval solutions when restoring causes through observed effects, the constraints in the form of linguistic modifiers are imposed on the measures of causes significances. The interval solutions are modeled by the complete crisp solutions, that is, the maximum solutions for the vectors of binary weights of the linguistic modifiers. The search for approximate solutions of the SFRE amounts to solving the optimization problem using the genetic algorithm. Due to the properties of the solution set, the genetic search for the lower and upper subsets is parallelized for each aggregating solution. The developed method makes it possible to simplify the search for the solution set based on the constraints on accuracy (interpretability) of the applied problem.

Key words: extended *max-min* fuzzy relation equations, solution set, minimal (maximal) solutions, interval solutions, constrained linguistic solutions

1. Introduction

Fuzzy relations and compositional rule of inference are widely used in the problems of inverse inference when restoring causes from observed effects (Di Nola et al. 1989; Peeva and Kyosev 2004). Solving the System of Fuzzy Relation Equations (SFRE) with *max-min* composition consists in finding the unique greatest solution and a set of minimal solutions (Di Nola et al. 1989; Peeva and Kyosev 2004). In (Markovskii 2005; Lin et al. 2011; Bartl and Belohlavek 2015), the connection between the problem of finding the set of minimal solutions and the covering problem, which is NP-hard, is proven. However, for applied problems, it is impractical to determine all minimal solutions (Bartl and Prochazka 2017). Moreover, for multivariable dependencies, cause-effect interconnections are modeled by the extended rule of inference (Yager and Filev 1994). In this case, one has to deal with the approximate solvability of the SFRE (Rotshtein and Rakytyanska 2011, 2012).

In (Rotshtein and Rakytyanska 2011, 2012), the genetic algorithm for solving the *max-min* SFRE was proposed to overcome NP-complexity. The method (Rotshtein and Rakytyanska 2011, 2012) is based on the approximate solvability of the SFRE with simplified and multilevel composition laws which describe cause-effect connections represented by fuzzy relations and rules. However, for a set of interval solutions, the problem of excessive granularity remains unresolved. This leads to unnecessary detail, which requires additional computational resources, but does not improve the interpretability of the obtained solutions. The paper (Peeva 2013) analyzes approaches to reducing the complexity of the problem of finding minimal solutions by decomposing the *max-min* SFRE and aggregating subsets of solutions; partitioning the search space and cutting off redundant search branches; studying the properties of solutions in the context of the applied problem. Therefore, it is important to develop the method (Rotshtein and Rakytyanska 2011, 2012) to simplify the search for the set of solutions necessary for solving the applied problems.

2. Literature Review

Despite the well-developed apparatus for solving the *max-min* SFRE, there are still problems of its application in practice due to the complexity of finding the set of minimal solutions. Therefore, in recent years, research has focused on developing methods aimed at simplifying the search for the solution set. In this case, the search algorithms are based on some new additional properties of the solution set.

Analytical methods involve the transformation of the SFRE into a characteristic matrix, where solutions are classified as attainable if they satisfy the equations of the system (Markovskii 2005; Lin et al. 2011). In (Markovskii 2005; Lin et al. 2011), it was shown that the minimal solutions of the SFRE correspond to the irredundant coverings of the characteristic matrix. In (Peeva 2013; Shieh 2013), the solvability conditions are formulated and algorithms for finding the minimal coverings of the matrix are developed. In (Peeva 2013), cutting off redundant branches of the search is carried out based on dominance relations that allow ordering the coefficients of the characteristic matrix which contribute to solvability of the system. Reducing the complexity of the search is achieved by decomposing the SFRE into the independent vector and matrix subsystems (Fan et al. 2020; Sun et al. 2020; Stankovic et al. 2017). When aggregating the subsets of solutions, the aggregation operators admit weakening the solvability conditions of the decomposed SFRE (Medina 2017).

The algorithms recommended in practice due to acceptable complexity are based on the assumption that for every solution there exists a minimum solution less than or equal to a given one (Yeh 2008; Shieh 2008). Following (Díaz-Moreno et al. 2017; Sun et al. 2016; Li and Wang 2021), other minimal solutions can be derived from their predecessors. Simplifying the search for a set of minimal solutions involves combining them to find the widest interval. This approach eliminates duplication of solutions when finding the complete solution set. A new way in solving the SFRE is the method (Turunen 2020), which establishes the conditions for the existence of the lower and upper bounds without finding the complete solution set. If general or global solutions do not exist, conditions for the existence of partial interval or point wise solutions are formulated, which does not require the analytical solvability of the SFRE.

In the context of the practical applications, the relationships between the significance measures of the decisions can be investigated. The solutions that represent the most profitable coverings are classified as strong (Yang X 2020). For many practical applications, the priority level of solutions is important. It is shown in (Yang et al. 2017; Qiu et al. 2021; Yang et al. 2018), that lexicographic order relations are satisfied for the set of minimal solutions. Reduction of the decision tree is associated with the ordering of the lexicographic solutions by moving from the set of solutions to the set of indices. Therefore, at the index level, there is a set of minimal lexicographic solutions. For solving the problems of minimization (maximization) of the objective function with a system of constraints in the form of the SFRE, the concepts of leximinimal (leximaximal) solutions are introduced, which are Pareto-optimal on the set of minimal solutions (Yang et al. 2018; Yang XP 2020; Guu and Wu 2019).

In (Bartl et al. 2012; Bartl and Trnecka 2021), the method for solving the SFRE with constraints imposed on the significance measures in the form of linguistic modifiers is proposed. If the measures of causes and effects significances as well as the fuzzy relations take values from a set, e.g., {0, 0.25, 0.5, 0.75, 1}, then the conditions of analytical solvability are generalized to the constrained solutions of the SFRE (Bartl et al. 2012). Such an approach implies the composition over modified fuzzy relations (Cao et al. 2018). It follows from analytical solvability, that the number of linguistic modifiers should be the same for all causes and effects (Sun and Qu 2021). When the constraints are empty, the linguistic solutions turn into the ordinary ones in the form of intervals (Bartl et al 2012). When finding all minimal solutions without taking into account the constrained scale, one needs to go through all combinations of the values of significance measures of the causes (Bartl and Trnecka 2021). On the other hand, the linguistic scale of truth values is sufficient for many applied problems, where the significance measures of the causes are described using the linguistic modifiers, e.g., *weak, moderate, strong increase* (Novák 2015; Le and Tran 2018; Vidal et al. 2020). As a result, finding all minimal solutions under the imposed constraints allows skipping a considerable amount of these combinations (Bartl and Trnecka 2021).

Thus, methods aimed at simplifying the search for minimal solutions rely on analytical solvability (and, in the absence of exact solutions, on partial solvability) of the SFRE with *max-min* composition (Peeva 2013; Shieh 2013). However, this type of composition is a special case and its use in practice is limited since multifactorial dependencies are modeled by the extended inference rule (Yager and Filev 1994). The methods of finding the widest intervals (Yeh 2008; Shieh 2008; Díaz-Moreno et al. 2017; Sun et al. 2016; Li and Wang 2021; Turunen 2020) and ranking the solutions (Yang et al. 2017; Qiu et al. 2021; Yang et al. 2018) abandon the structure of the solution set in the form of a set of explanations which is classical for the problems of inverse inference. Such methods make it possible to obtain only the absolute range of changes in factors or their relative significance, which leads to the loss of relationships between factors. To achieve the accuracy of the inverse inference, it is necessary to tune the fuzzy model to experimental data (Rotshtein and Rakytyanska 2011, 2012). Moreover, when building the constrained solutions (Bartl et al. 2012; Bartl and Trnecka 2021), the significance measures associated with linguistic modifiers are also subject to tuning. However, since the method of constrained solutions (Bartl et al. 2012; Bartl and Trnecka 2021) relies on analytical solvability, the requirements imposed on the significance measures of causes, effects and fuzzy relations cannot be satisfied when tuning the model. In practice, solving the extended SFRE for multifactorial dependencies with tuning to experimental data remains an unresolved problem. Besides, preserving the properties of the solution set when reducing the search area remains the crucial issue that seriously limits the use of fuzzy relational calculus in solving inverse problems.

3. Problem Statement

To ensure practical application, the decisive aspects of the method for solving the SFRE are the extended type of composition; approximate solvability; complexity reduction while preserving the properties of the solution set; interpretable solutions; tuning solutions to experimental data.

This work is the development of the method of inverse inference (Rakityanskaya and Rotshtein 2007; Rotshtein and Rakytyanska 2009), which consists in solving the *max-min* SFRE with simultaneous tuning the fuzzy model to experimental data. For the extended *max-*

min composition (Yager and Filev 1994), the subsets of solutions which correspond to subsystems described by fuzzy relations and rules were investigated in (Rotshtein and Rakytyanska 2011, 2012, 2014). It was shown in (Rakytyanska 2018, 2023), that the aggregation of the subsystems is carried out using the *max-min* or dual *min-max* composition. Therefore, solving the extended SFRE implies search for all minimal and maximal solutions (Peeva and Kyosev 2004). However, in the general case, the problem remains unresolved, since there is not a single, but a set of solutions for aggregating the subsystems, which, in turn, yield new minimal (maximal) solutions.

The search for approximate solutions of the SFRE amounts to solving the optimization problem using the genetic algorithm (Rakityanskaya and Rotshtein 2007; Rotshtein and Rakytyanska 2009). The set of solutions is formed by repeated runs of the genetic algorithm until all solutions are found. As a result, finding all minimal (maximal) solutions leads to unjustified computational costs because of excessive granularity of solutions since the mechanisms for linguistic interpretation of the obtained intervals are absent. At the same time, in many real applications, the granularity of solutions is known in advance (Azarov et al 2021).

In this work, the method of inverse inference based on interpretable solutions of the extended *max-min* SFRE is developed. The interval and linguistic solutions with granular and relational structure of the solution set are investigated. Following (Bartl et al. 2012; Bartl and Trnecka 2021), to avoid excessive granularity and ensure interpretability of the interval solutions, the constraints in the form of linguistic modifiers are imposed on the measures of causes significances. Given the ranges of variation of the input parameters for each linguistic modifier, the permissible values of significance measures are defined with the help of the membership functions. The interval solutions are modeled using crisp solutions for the weights of linguistic modifiers. To cover the set of intervals, the concept of a complete crisp solution is introduced as the maximum solution for the vector of binary weights in the linguistic description of the interval solution (Rakytyanska 2023). Thus, the set of interval solutions is replaced by a reduced set of complete crisp solutions, that preserves the properties of the solution set in the form of a set of explanations (Rotshtein and Rakytyanska 2014; Rakytyanska 2018). The accuracy of linguistic solutions is ensured by tuning the membership functions of causes and effects and fuzzy relations using experimental data (Rotshtein and Rakytyanska 2011, 2012).

Following (Rotshtein and Rakytyanska 2011, 2012), the genetic algorithm for solving the SFRE is developed based on the properties of a set of ordinary and constrained solutions.

Since the distributive set theoretic law is satisfied for *max* and *min* operators, the extended *max-min* SFRE can be represented in the form of subsystems with *max-min* composition, which are aggregated using the *min* operator, or subsystems with dual *min-max* composition, which are aggregated using the *max* operator. In the general case, the solution set is constructed for a set of aggregating solutions. When decomposing the subsystems with *max-min* and dual *min-max* composition, the set of solutions can be decomposed into the lower and upper subsets bounded by the same aggregating solutions. Each lower (upper) subset is defined by the unique greatest (least) or aggregating solution and a set of minimal (maximal) solutions. It follows from the properties of the set of ordinary solutions that the set of constrained solutions is decomposed into the lower and upper subsets of complete crisp solutions bounded by the constrained aggregating solutions.

The properties of the solution set allow us to parallelize the genetic search for the lower and upper subsets bounded by the same aggregating solutions. At first, the set of ordinary (constrained) aggregating solutions is found. Then, the pool of optimization problems is distributed for finding all lower and upper subsets simultaneously. The search for the minimal (maximal) solutions for each subset is carried out by repeated runs of the genetic algorithm (Rotshtein and Rakytyanska 2011, 2012). Following (Bartl et al. 2012; Bartl and Trnecka 2021), the computation time is shortened due to reduction of the number of constrained aggregating solutions and minimal (maximal) solutions. Thus, the developed method makes it possible to simplify the search for the solution set of the extended *max-min* SFRE based on the constraints on accuracy (interpretability) of the applied problem.

The aim of the work is to develop the method of inverse inference based on interpretable constrained solutions of the extended *max-min* SFRE. The imposed constraints should simplify the process of numerically finding the complete solution set. To achieve this aim, the following objectives were accomplished: to investigate the properties of the set of interval (constrained) solutions; to develop the genetic algorithm for solving the SFRE.

4 Method of inverse inference based on solving the extended *max-min* SFRE

4.1 Extended *max-min* SFRE

The object with inputs $x_i \in [\underline{x}_i, \overline{x}_i]$, $i = 1, \dots, n$, and outputs $y_j \in [\underline{y}_j, \overline{y}_j]$, $j = 1, \dots, m$, is considered. Fuzzy terms e_{jp}, $p = 1, \dots, q_j$, for estimating the variables y_j, $j = 1, \dots, m$, are associated with the observed effects. Fuzzy terms c_{il}, $l = 1, \dots, k_i$, for estimating the variables x_i, $i = 1, \dots, n$, correspond to the causes of the observed effects. The problem of inverse inference is to restore the causes (inputs) through the observed effects (outputs).

The cause-effect dependency can be described with the help of the extended *max-min* rule of inference (Yager and Filev 1994):

$$\boldsymbol{\mu}^{F_j} = \boldsymbol{\mu}^{A_1} \circ \boldsymbol{R}_{1j} \cap \ldots \cap \boldsymbol{\mu}^{A_n} \circ \boldsymbol{R}_{nj}, j = 1 \ldots, m, \quad (1)$$

where $\boldsymbol{\mu}^{A_i} = (\mu^{c_{i1}}, \ldots, \mu^{c_{ik_i}})$ is the vector of causes c_{il}, $i = 1, \ldots, n$, $l = 1, \ldots, k_i$, significance measures;

$\boldsymbol{\mu}^{F_j} = (\mu^{e_{j1}}, \ldots, \mu^{e_{jq_j}})$ is the vector of effects e_{jp}, $j = 1, \ldots, m$, $p = 1, \ldots, q_j$, significance measures;

$\boldsymbol{R}_{ij} \subseteq c_{il} \times e_{jp}$=$[r_{il,jp}]$ is the fuzzy relation matrix;

$(\circ, \cap)$ is the operation of extended *max-min* composition (Yager and Filev 1994).

We define the membership functions $\mu^{c_{il}}(x_i)$ and $\mu^{e_{jp}}(y_j)$ to find the measures of causes and effects significances for the given values of the input and output variables.

Let us redenote:

$\boldsymbol{\mu}^{C}$=$(\mu^{C_1}, \ldots, \mu^{C_N}) = (\boldsymbol{\mu}^{A_1}, \ldots, \boldsymbol{\mu}^{A_n})$ is the vector of causes C_I significance measures, where $N = k_1 + \ldots + k_n$;

$\boldsymbol{\mu}^{E}$=$(\mu^{E_1}, \ldots, \mu^{E_M}) = (\boldsymbol{\mu}^{F_1}, \ldots, \boldsymbol{\mu}^{F_m})$ is the vector of effects E_J significance measures, where $M = q_1 + \ldots + q_m$.

The extended SFRE (1) can be represented in the form of subsystems with *max-min* composition, which are aggregated using the *min* operator (Yager and Filev 1994):

$$\mu_i^{E_J} = \vee_{l=1}^{k_i} (\mu^{c_{il}} \wedge r_{il,J}), i = 1, \ldots, n, \quad (2)$$

$$\mu^{E_J} = \wedge_{i=1}^{n} (\mu_i^{E_J}), J = 1, \ldots, M. \quad (3)$$

Here $\mu_i^{E_J}$ is the measure of the effect E_J significance for the i-*th* subsystem.

Since the distributive set theoretic law is satisfied for the *max* and *min* operators, the extended SFRE (1) can be rearranged in the form of subsystems with dual *min-max* composition, which are aggregated using the *max* operator (Rotshtein and Rakytyanska 2011, 2012):

$$\mu^{H_L} = \wedge_{i=1}^{n} (\mu^{a_L^i}), L = 1, \ldots, N^*, \quad (4)$$

$$\mu^{E_J} = \vee_{L=1}^{N^*} (\mu_J^{H_L}) = \vee_{L=1}^{N^*} (\mu^{H_L} \wedge r_{LJ}^*), J = 1, \ldots, M. \quad (5)$$

Here $a_L^i \in \{c_{i1}, \ldots, c_{ik_i}\}$ is the fuzzy term for estimating the variable x_i in the combination of causes H_L;

N^* is the number of the combinations of causes;

μ^{H_L} is the significance measure of the combination of causes H_L;

r_{LJ}^* is the combined relation $H_L \times E_J$;

$\mu_J^{H_L}$ is the significance measure of the combination of causes H_L for the effect E_J.

4.2 Structure of the constrained linguistic solution

We shall denote:

$\{\alpha_{I1},\ldots,\alpha_{Ig_I}\}$ is the set of linguistic modifiers for estimating the significance measure μ^{C_I}, $I = 1, \ldots, N$;

$\underline{\mu}_K^{C_I}$ ($\overline{\mu}_K^{C_I}$) is the lower (upper) bound of the significance measure μ^{C_I} for the linguistic modifier α_{IK}, $I = 1, \ldots, N, K = 1, \ldots, g_I$.

It is supposed that for each linguistic solution $\mu^{C_I} = \alpha_{IK}$, $I = 1, \ldots, N$, $K = 1, \ldots, g_I$, the range of variation of the input parameter $x_i(\alpha_{IK})$ =[$\underline{x}_{IK}$, $\overline{x}_{IK}$], $\underline{x}_{IK}$=$\overline{x}_{I,K-1}$, is known. In this case, the bounds of intervals of the significance measures $\mu^{C_I}(\alpha_{IK}) = [\underline{\mu}_K^{C_I}, \overline{\mu}_K^{C_I}]$, $\underline{\mu}_K^{C_I} = \overline{\mu}_{K-1}^{C_I}$, associated with the linguistic modifiers α_{IK}, can be obtained with the help of the membership functions of the fuzzy terms C_I.

Following (Bartl et al. 2012), the constraints on the values of the significance measures μ^{C_I} are imposed as follows:

$$\mu^{C_I} \in \{\overline{\mu}_1^{C_I},\ldots, \overline{\mu}_K^{C_I},\ldots,1\}, I = 1, \ldots, N, K = 1, \ldots, g_I. \quad (6)$$

Let $\boldsymbol{\mu}^C = (\mu^{C_1},\ldots,\mu^{C_N})$, $\mu^{C_I} \in [\underline{\mu}^{C_I}, \overline{\mu}^{C_I}]$, be the interval solution of the SFRE (1), where $\underline{\mu}^{C_I}$ ($\overline{\mu}^{C_I}$) is the lower (upper) bound of the significance measure μ^{C_I}.

Given constraints (6), the structure of the linguistic solution is defined as follows (Rakytyanska 2023). We shall denote:

$\boldsymbol{W}_I = (w_{I1},\ldots,w_{Ig_I})$, $I = 1, \ldots, N$, is the vector of weights of linguistic modifiers, where $w_{IK} = 1$, if the upper bound $\overline{\mu}_K^{C_I}$ is the solution of the SFRE (1); $w_{IK} = 0$ otherwise.

Then, the interval solution can be replaced by the set of explanations:

$$\mu^{C_I}=\{\alpha_{I1} \text{ (with weight } w_{I1}) \text{ OR } \ldots \text{ OR } \alpha_{Ig_I} \text{ (with weight } w_{Ig_I})\}, I = 1, \ldots, N, \quad (7)$$

where $w_{IK} = 1(0)$ if the modifier α_{IK} is present (absent) in the linguistic description of the interval solution $\mu^{C_I} \in [\underline{\mu}^{C_I}, \overline{\mu}^{C_I}]$.

The following *max-min* SFRE, which is derived from the relation (7), connects the significance measures for the interval and constrained solutions (Rakytyanska 2023):

$$\mu^{C_I} = \vee_{K=1}^{g_I}(\overline{\mu}_K^{C_I} \wedge w_{IK}), I = 1, \ldots, N. \quad (8)$$

Following (Di Nola et al. 1989; Peeva and Kyosev 2004), the unique maximal solution of the SFRE (8) completely covers the interval solution $[\underline{\mu}^{C_I}, \overline{\mu}^{C_I}]$, $I = 1, \ldots, N$, and the set of minimal solutions corresponds to subintervals of $[\underline{\mu}^{C_I}, \overline{\mu}^{C_I}]$ to be merged as redundant.

The unique maximal solution $\overline{\boldsymbol{W}}_I = (\overline{w}_{I1}, \ldots, \overline{w}_{Ig_I})$, $I = 1, \ldots, N$, where $\overline{w}_{IK}$ are the upper bounds for the weights of linguistic modifiers α_{IK}, is called a complete crisp solution (Rakytyanska 2023).

4.3 Optimization problems for solving the extended *max-min* SFRE

Following (Rotshtein and Rakytyanska 2011, 2012), the problem of solving the SFRE (1) is formulated as follows. The vector of causes significance measures $\boldsymbol{\mu}^C = (\mu^{C_1}, \ldots, \mu^{C_N})$, $\mu^{C_I} \in [0, 1]$, $I = 1, \ldots, N$, should be found which provides the least distance between the observed and model measures of effects significances:

$$\Delta(\boldsymbol{\mu}^C) = \sum_{J=1}^{M}\left[\mu^{E_J}(y_j) - \mu^{E_J}(\mu^{C_1}, \ldots, \mu^{C_N})\right]^2 = \min_{\boldsymbol{\mu}^C}. \quad (9)$$

Statement 1. The set S of interval solutions of the SFRE (1) is defined by the set of aggregating solutions $\hat{G} = \{\hat{\boldsymbol{\mu}}_p^C,\ p = 1, \ldots, \hat{Z}\}$, where for each solution $\hat{\boldsymbol{\mu}}_p^C \in \hat{G}$ there exist the set of minimal solutions $\underline{B}_p = \{\underline{\boldsymbol{\mu}}_{pl}^C,\ l = 1, \ldots, \underline{Z}_p\}$ and the set of maximal solutions $\overline{B}_p = \{\overline{\boldsymbol{\mu}}_{ph}^C,\ h = 1, \ldots, \overline{Z}_p\}$:

$$S = \bigcup_{\hat{\boldsymbol{\mu}}_p^C \in \hat{G}} \bigcup_{\underline{\boldsymbol{\mu}}_{pl}^C \in \underline{B}_p} \bigcup_{\overline{\boldsymbol{\mu}}_{ph}^C \in \overline{B}_p} \left[[\underline{\boldsymbol{\mu}}_{pl}^C, \hat{\boldsymbol{\mu}}_p^C] \cup [\hat{\boldsymbol{\mu}}_p^C, \overline{\boldsymbol{\mu}}_{ph}^C]\right]. \quad (10)$$

Here $\hat{\boldsymbol{\mu}}_p^C = (\hat{\mu}_p^{C_1}, \ldots, \hat{\mu}_p^{C_N})$ is the vector of aggregation of causes significance measures;

$\underline{\boldsymbol{\mu}}_{pl}^C = (\underline{\mu}_{pl}^{C_1}, \ldots, \underline{\mu}_{pl}^{C_N})$ and $\overline{\boldsymbol{\mu}}_{ph}^C = (\overline{\mu}_{ph}^{C_1}, \ldots, \overline{\mu}_{ph}^{C_N})$ are the vectors of lower and upper bounds of causes significance measures.

Proof. The formula (10) follows from the properties of the solution set of the *max-min* and dual *min-max* SFRE (Di Nola et al. 1989; Peeva and Kyosev 2004).

Since the SFRE (1) contains subsystems (2) with *max*-*min* composition and aggregation of the subsystems is carried out using the *min* operator, then the solution set S has the lower subsets $\underline{S}_p$, $p = 1, \dots, \hat{Z}^1$, each of which is defined by the unique maximal solution $\overline{\boldsymbol{\mu}}_p^{1C}$ and the set of minimal solutions $\underline{B}_p = \{\underline{\boldsymbol{\mu}}_{pl}^C,\ l = 1, \dots, \underline{Z}_p\}$:

$$\underline{S}_p = \bigcup_{\underline{\boldsymbol{\mu}}_{pl}^C \in \underline{B}_p} \left[\underline{\boldsymbol{\mu}}_{pl}^C,\ \overline{\boldsymbol{\mu}}_p^{1C}\right],\ p = 1, \dots, \hat{Z}^1.$$

Here $\overline{\boldsymbol{\mu}}_p^{1C} = (\overline{\mu}_p^{1C_1}, \dots, \overline{\mu}_p^{1C_N})$ is the vector of upper bounds of causes significance measures for the lower subset $\underline{S}_p$.

On the other hand, since the SFRE (1) contains subsystems (4) with dual *min*-*max* composition and aggregation of the subsystems is carried out using the *max* operator, then the solution set S has the upper subsets $\overline{S}_p$, $p = 1, \dots, \hat{Z}^2$, each of which is defined by the unique minimal solution $\underline{\boldsymbol{\mu}}_p^{2C}$ and the set of maximal solutions $\overline{B}_p = \{\overline{\boldsymbol{\mu}}_{ph}^C,\ h = 1, \dots, \overline{Z}_p\}$:

$$\overline{S}_p = \bigcup_{\overline{\boldsymbol{\mu}}_{ph}^C \in \overline{B}_p} \left[\underline{\boldsymbol{\mu}}_p^{2C},\ \overline{\boldsymbol{\mu}}_{ph}^C\right],\ p = 1, \dots, \hat{Z}^2.$$

Here $\underline{\boldsymbol{\mu}}_p^{2C} = (\underline{\mu}_p^{2C_1}, \dots, \underline{\mu}_p^{2C_N})$ is the vector of lower bounds of causes significance measures for the upper subset $\overline{S}_p$.

If $\boldsymbol{\mu}^C \in \underline{S}_p$, where $\mu^{C_I} \le \overline{\mu}_p^{1C_I}$, then $\mu_i^{E_J}(\boldsymbol{\mu}^C) = \mu_i^{E_J}(\overline{\boldsymbol{\mu}}_p^{1C})$, $i = 1, \dots, n$, and if $\boldsymbol{\mu}^C \in \overline{S}_p$, where $\mu^{C_I} \ge \underline{\mu}_p^{2C_I}$, then $\mu_J^{H_L}(\boldsymbol{\mu}^C) = \mu_J^{H_L}(\underline{\boldsymbol{\mu}}_p^{2C})$, $L = 1, \dots, N^*$.

It follows from the equality of the right-hand sides of the equations (3) and (5) that the set of upper solutions $\hat{G}^1 = \{\overline{\boldsymbol{\mu}}_p^{1C},\ p = 1, \dots, \hat{Z}^1\}$ of the lower subsets $\underline{S}_p$, and the set of lower solutions $\hat{G}^2 = \{\underline{\boldsymbol{\mu}}_p^{2C},\ p = 1, \dots, \hat{Z}^2\}$ of the upper subsets $\overline{S}_p$ are equal, that is $\hat{Z}^1 = \hat{Z}^2 = \hat{Z}$ and $\overline{\boldsymbol{\mu}}_p^{1C} = \underline{\boldsymbol{\mu}}_p^{2C} = \hat{\boldsymbol{\mu}}_p^C$, $p = 1, \dots, \hat{Z}$. Thus, the set $\hat{G}^1 = \hat{G}^2 = \hat{G} = \{\hat{\boldsymbol{\mu}}_p^C,\ p = 1, \dots, \hat{Z}\}$ is the set of aggregating solutions.

Then, by performing the union $\bigcup_{\hat{\boldsymbol{\mu}}_p^C \in \hat{G}} (\underline{S}_p \cup \overline{S}_p)$, we obtain the formula (10).

We shall redenote the vector of weights of linguistic modifiers as $\boldsymbol{V} = (\boldsymbol{W}_1, \dots, \boldsymbol{W}_N) = (v_1, \dots, v_T)$, where $T = g_1 + \dots + g_N$.

Given constraints (6), the problem of solving the SFRE (1), (8) is formulated as follows (Rakytyanska 2023). The vector of weights of linguistic modifiers $\boldsymbol{V} = (\boldsymbol{W}_1, \dots, \boldsymbol{W}_N) = (v_1, \dots, v_T)$, $v_P \in \{0,\ 1\}$, should be found which provides the least distance between the observed and model measures of effects significances:

$$\Delta(\boldsymbol{V}) = \sum_{J=1}^{M} \left[\mu^{E_J}(y_j) - \mu^{E_J}(\boldsymbol{W}_1, \dots, \boldsymbol{W}_N)\right]^2 = \min_{\boldsymbol{V}}. \tag{11}$$

Let us introduce the following concepts:

$\widehat{\boldsymbol{V}} = (\widehat{\boldsymbol{W}}_1, \dots, \widehat{\boldsymbol{W}}_N) = (\hat{v}_1, \dots, \hat{v}_T)$ is the constrained aggregating solution which is defined by the set of aggregating indices $\{\hat{X}_1, \dots, \hat{X}_N\}$, where $\widehat{w}_{IK} = 1$, if $K = \hat{X}_I$;

$\overleftarrow{\boldsymbol{V}} = (\overleftarrow{\boldsymbol{W}}_1, \dots, \overleftarrow{\boldsymbol{W}}_N) = (\overleftarrow{v}_1, \dots, \overleftarrow{v}_T)$ is the upper bounded complete crisp solution, where for all $\overleftarrow{w}_{IK} = 1$, $K \leq \hat{X}_I$;

$\vec{\boldsymbol{V}} = (\vec{\boldsymbol{W}}_1, \dots, \vec{\boldsymbol{W}}_N) = (\vec{v}_1, \dots, \vec{v}_T)$ is the lower bounded complete crisp solution, where for all $\vec{w}_{IK} = 1$, $K \geq \hat{X}_I$.

Statement 2. The set S_c of constrained solutions of the SFRE (1), (8) is defined by the set of constrained aggregating solutions $\widehat{D} = \{\widehat{\boldsymbol{V}}_p,\ p = 1, \dots, \hat{Q}\}$, where for each solution $\widehat{\boldsymbol{V}}_p \in \widehat{D}$ there exist the lower and upper subsets of complete crisp solutions $\underline{D}_p = \{\overleftarrow{\boldsymbol{V}}_{pl},\ l = 1, \dots, \underline{Q}_p\}$ and $\overline{D}_p = \{\vec{\boldsymbol{V}}_{ph},\ h = 1, \dots, \overline{Q}_p\}$ with the set of aggregating indices $\{\hat{X}_1^p, \dots, \hat{X}_N^p\}$:

$$S_c = \bigcup_{\widehat{\boldsymbol{V}}_p \in \widehat{D}} \ \bigcup_{\overleftarrow{\boldsymbol{V}}_{pl} \in \underline{D}_p} \ \bigcup_{\vec{\boldsymbol{V}}_{ph} \in \overline{D}_p} (\overleftarrow{\boldsymbol{V}}_{pl} \cup \vec{\boldsymbol{V}}_{ph}). \tag{12}$$

Here $\widehat{\boldsymbol{V}}_p = (\widehat{\boldsymbol{W}}_1^p, \dots, \widehat{\boldsymbol{W}}_N^p) = (\hat{v}_1^p, \dots, \hat{v}_T^p)$ is the constrained aggregating solution for the lower and upper subsets $\underline{D}_p$ and $\overline{D}_p$, where $\widehat{w}_{IK}^p = 1$, if $K = \hat{X}_I^p$;

$\overleftarrow{\boldsymbol{V}}_{pl} = (\overleftarrow{\boldsymbol{W}}_1^{pl}, \dots, \overleftarrow{\boldsymbol{W}}_N^{pl}) = (\overleftarrow{v}_1^{pl}, \dots, \overleftarrow{v}_T^{pl})$ is the upper bounded complete crisp solution for the lower subset $\underline{D}_p$, where for all $\overleftarrow{w}_{IK}^{pl} = 1$, $K \leq \hat{X}_I^p$;

$\vec{\boldsymbol{V}}_{ph} = (\vec{\boldsymbol{W}}_1^{ph}, \dots, \vec{\boldsymbol{W}}_N^{ph}) = (\vec{v}_1^{ph}, \dots, \vec{v}_T^{ph})$ is the lower bounded complete crisp solution for the upper subset $\overline{D}_p$, where for all $\vec{w}_{IK}^{ph} = 1$, $K \geq \hat{X}_I^p$.

Proof. The formula (12) follows from the properties (10) of the solution set of the extended *max-min* SFRE.

4.4 Genetic algorithm for solving the extended *max-min* SFRE

Due to the properties of the solution set of the extended *max-min* SFRE, finding the set of interval solutions is reduced to parallel genetic search for the lower and upper subsets $\underline{S}_p$ and $\overline{S}_p$ bounded by the same aggregating solutions $\hat{\boldsymbol{\mu}}_p^C \in \hat{G},\ p = 1, \dots, \hat{Z}$. In the case of constrained solutions, the parallel genetic search is performed to find the lower and upper subsets $\underline{D}_p$ and $\overline{D}_p$ separated by the aggregating solutions $\widehat{\boldsymbol{V}}_p \in \widehat{D}, p = 1, \dots, \hat{Q}$.

The genetic algorithm is performed in two stages: search for the set of aggregating solutions $\hat{G} = \{\hat{\boldsymbol{\mu}}_p^C,\ p = 1, \dots, \hat{Z}\}$; parallel search for the sets $\underline{B}_p$ ($\overline{B}_p$) of lower (upper) bounds for each aggregating solution $\hat{\boldsymbol{\mu}}_p^C \in \hat{G}$. In the case of constrained solutions, formation of the set of aggregating solutions $\widehat{D} = \{\widehat{\boldsymbol{V}}_p,\ p = 1, \dots, \hat{Q}\}$ precedes the parallel search for the lower (upper) subsets of complete crisp solutions $\underline{D}_p$ ($\overline{D}_p$).

When searching for the aggregating solutions, the chromosome encodes the solutions $\boldsymbol{\mu}^C = (\mu^{C_1}, \dots, \mu^{C_N})$ or $\boldsymbol{V} = (\boldsymbol{W}_1, \dots, \boldsymbol{W}_N)$. The cross-over operation consists in exchanging parts of the chromosomes inside each solution μ^{C_I} or $\boldsymbol{W}_I$, $I = 1, \dots, N$. The fitness function is based on the criterion (9) or (11).

The aggregating solution is formed by a stepwise increment (decrement) until the upper solution of the lower subset and the lower solution of the upper subset coincide. The set of aggregating solutions is formed by repeated runs of the genetic algorithm if new aggregating solutions are found. Given constraints, the weights are activated by a stepwise increment (decrement) until the set of upper aggregating indices for the lower subset and the set of lower aggregating indices for the upper subset coincide. To form the set of constrained aggregating solutions, the genetic algorithm is repeatedly run if new sets of aggregating indices are found.

The criterion for stopping the genetic algorithm is the absence of new aggregating solutions or new sets of aggregating indices within a given number of iterations.

Let $\boldsymbol{\mu}^C(t) = (\mu^{C_1}(t), \dots, \mu^{C_N}(t))$ be some *t*-th solution of the optimization problem (9);

$\boldsymbol{V}(t) = (\boldsymbol{W}_1(t), \dots, \boldsymbol{W}_N(t))$ be some *t*-th solution of the optimization problem (11) with some set of aggregating indices $\{X_1(t), \dots, X_N(t)\}$.

Algorithm 1: Formation of the set of aggregating solutions

1: **while** [new aggregating solutions are found] **do**

2: Search for the null solution $\boldsymbol{\mu}_0^C = (\mu_0^{C_1}, \dots, \mu_0^{C_N})$ of the optimization problem (9)

3: // Define the search space for the aggregating solution $\hat{\boldsymbol{\mu}}_p^C$

if $\mu_0^{C_I} \in \underline{S}_p$, then $\hat{\mu}_p^{C_I} \in [\mu_0^{C_I},\ 1]$; if $\mu_0^{C_I} \in \overline{S}_p$, then $\hat{\mu}_p^{C_I} \in [0,\ \mu_0^{C_I}]$

Exclude previous solutions $\hat{\boldsymbol{\mu}}_k^C$, $k < p$, from the search space

while [new bounds of the interval are found] **do** // Stepwise increment (decrement)

Go to $\boldsymbol{\mu}^C\ (t) = (\mu^{C_1}(t), \dots, \mu^{C_N}(t))$, where $\Delta(\boldsymbol{\mu}^C\ (t)) = \Delta(\boldsymbol{\mu}_0^C)$

if $\mu^{C_I}(t) \geq \mu^{C_I}(t-1)$ and $\mu_i^{E_J}(\mu^{C_I}(t)) = \mu_i^{E_J}(\mu^{C_I}(t-1))$, then $\mu^{C_I}(t) \in \underline{S}_p$

if $\mu^{C_I}(t) \leq \mu^{C_I}(t-1)$ and $\mu_J^{H_L}(\mu^{C_I}(t)) = \mu_J^{H_L}(\mu^{C_I}(t-1))$, then $\mu^{C_I}(t) \in \overline{S}_p$

// Define the aggregating solution

if $\boldsymbol{\mu}^C\ (t) \neq \boldsymbol{\mu}^C\ (t-1)$, then $\hat{\mu}_p^{C_I}$=$\mu^{C_I}(t)$

if $\boldsymbol{\mu}^C\ (t) = \boldsymbol{\mu}^C\ (t-1)$, then the search is stopped

end while [new bounds of the interval are found]

end while [new aggregating solutions are found]

Algorithm 2: Formation of the set of constrained aggregating solutions

while [new sets of aggregating indices are found] **do**

Search for the null solution $\boldsymbol{V}_0 = (\boldsymbol{W}_1^0, \dots, \boldsymbol{W}_N^0)$ of the optimization problem (11)

Define the null set of aggregating indices $\{X_1^0, \dots, X_N^0\}$

// Define the search space for the aggregating indices $\hat{X}_I^p$ of the solutions $\widehat{\boldsymbol{V}}_p$

if $\boldsymbol{W}_I^0 \in \underline{D}_p$, then $\hat{X}_I^p \in \{X_I^0,\ g_I\}$; if $\boldsymbol{W}_I^0 \in \overline{D}_p$, then $\hat{X}_I^p \in \{1,\ X_I^0\}$

Exclude previous solutions $\widehat{\boldsymbol{V}}_k$, $k < p$, from the search space

while [new aggregating bounds are found] **do** // Stepwise increment (decrement)

Go to $\boldsymbol{V}(t) = (\boldsymbol{W}_1(t), \dots, \boldsymbol{W}_N(t))$, where $\Delta(\boldsymbol{V}(t)) = \Delta(\boldsymbol{V}_0)$

if $X_I(t) \geq X_I(t-1)$ and $\mu_i^{E_J}(\boldsymbol{W}_I(t)) = \mu_i^{E_J}(\boldsymbol{W}_I(t-1))$, then $\boldsymbol{W}_I(t) \in \underline{D}_p$

if $X_I(t) \leq X_I(t-1)$ and $\mu_J^{H_L}(\boldsymbol{W}_I(t)) = \mu_J^{H_L}(\boldsymbol{W}_I(t-1))$, then $\boldsymbol{W}_I(t) \in \overline{D}_p$

// Define the constrained aggregating solution

if $\boldsymbol{V}(t) \neq \boldsymbol{V}(t-1)$, that is $X_I(t) \neq X_I(t-1)$, then $\widehat{\boldsymbol{V}}_p$=$\boldsymbol{V}(t)$ and $\hat{X}_I^p = X_I(t)$

if $\boldsymbol{V}(t) = \boldsymbol{V}(t-1)$, that is $X_I(t) = X_I(t-1)$, then the search is stopped

end while [new aggregating bounds are found]

end while [new sets of aggregating indices are found]

When searching for the lower and upper bounds, the chromosome is distributed to solutions $\boldsymbol{\mu}^{1C} = (\mu^{1C_1}, \dots, \mu^{1C_N})$ and $\boldsymbol{\mu}^{2C} = (\mu^{2C_1}, \dots, \mu^{2C_N})$ for the lower and upper subsets $\underline{S}_p$

and $\overline{S}_p$. In the case of constrained solutions, the chromosome is distributed to solutions $\boldsymbol{V}^1 = (\boldsymbol{W}_1^1, \ldots, \boldsymbol{W}_N^1)$ and $\boldsymbol{V}^2 = (\boldsymbol{W}_1^2, \ldots, \boldsymbol{W}_N^2)$ for the lower and upper subsets $\underline{D}_p$ and $\overline{D}_p$.

Formation of the lower and upper subsets is accomplished by way of solving a pool of the optimization problems (9) or (11). The interval solution is formed by a stepwise increment (decrement) until the widest interval is obtained. The set of intervals is formed by repeated runs of the genetic algorithm if new minimum (maximum) solutions are found. To cover the interval in the case of constraints, the stepwise increment (decrement) is performed until the maximum number of weights is activated. To cover the set of intervals, the genetic algorithm is repeatedly run if new complete crisp solutions are found.

The criterion for stopping the genetic algorithm is the absence of new lower (upper) bounds or new complete crisp solutions within a given number of iterations.

Let $\boldsymbol{\mu}^{1C}(t) = (\mu^{1C_1}(t), \ldots, \mu^{1C_N}(t))$ and $\boldsymbol{\mu}^{2C}(t) = (\mu^{2C_1}(t), \ldots, \mu^{2C_N}(t))$ be some *t*-th solutions of the optimization problem (9) for the lower and upper subsets;

$\boldsymbol{V}^1(t) = (\boldsymbol{W}_1^1(t), \ldots, \boldsymbol{W}_N^1(t))$ and $\boldsymbol{V}^2(t) = (\boldsymbol{W}_1^2(t), \ldots, \boldsymbol{W}_N^2(t))$ be some *t*-th solutions of the optimization problem (11) for the lower and upper subsets.

Algorithm 3: Formation of the set of lower and upper bounds

while [new lower and upper solutions are found] **do**

  Search for the null solutions of the optimization problem (9)

  $\boldsymbol{\mu}_0^{1C} = (\mu_0^{1C_1}, \ldots, \mu_0^{1C_N})$, $\mu_0^{1C_1} \le \hat{\mu}_p^{C_I}$, for the lower subset $\underline{S}_p$

  $\boldsymbol{\mu}_0^{2C} = (\mu_0^{2C_1}, \ldots, \mu_0^{2C_N})$, $\mu_0^{2C_I} \ge \hat{\mu}_p^{C_I}$, for the upper subset $\overline{S}_p$

  // Define the search space for the lower (upper) bounds $\underline{\boldsymbol{\mu}}_{pl}^{C}$ ($\overline{\boldsymbol{\mu}}_{ph}^{C}$)

  $\underline{\mu}_{pl}^{C_I} \in [0, \mu_0^{1C_I}]$; $\overline{\mu}_{ph}^{C_I} \in [\mu_0^{2C_I}, 1]$

  Exclude previous solutions $\underline{\boldsymbol{\mu}}_{pk}^{C}$, $k < l$, and $\overline{\boldsymbol{\mu}}_{pk}^{C}$, $k < h$, from the search space

  while [new bounds of the interval are found] **do** // Stepwise increment (decrement)

    Go to $\boldsymbol{\mu}^{1C}(t) = (\mu^{1C_1}(t), \ldots, \mu^{1C_N}(t))$, $\mu^{1C_I}(t) \le \mu^{1C_I}(t-1)$

    if $\Delta\left(\boldsymbol{\mu}^{1C}(t)\right) = \Delta(\boldsymbol{\mu}_0^{1C})$, then $\boldsymbol{\mu}^{1C} \in \underline{S}_p$

    Go to $\boldsymbol{\mu}^{2C}(t) = (\mu^{2C_1}(t), \ldots, \mu^{2C_N}(t))$, $\mu^{2C_I}(t) \ge \mu^{2C_I}(t-1)$

    if $\Delta\left(\boldsymbol{\mu}^{2C}(t)\right) = \Delta(\boldsymbol{\mu}_0^{2C})$, then $\boldsymbol{\mu}^{2C} \in \overline{S}_p$

    // Define the lower (upper) bounds

    if $\boldsymbol{\mu}^{1C}(t) \ne \boldsymbol{\mu}^{1C}(t-1)$, then $\underline{\mu}_{pl}^{C_I} = \mu^{1C_I}(t)$ for the lower subset $\underline{S}_p$

    if $\boldsymbol{\mu}^{2C}(t) \ne \boldsymbol{\mu}^{2C}(t-1)$, then $\overline{\mu}_{ph}^{C_I} = \mu^{2C_I}(t)$ for the upper subset $\overline{S}_p$

if $\boldsymbol{\mu}^{1C}(t) = \boldsymbol{\mu}^{1C}(t-1)$ and $\boldsymbol{\mu}^{2C}(t) = \boldsymbol{\mu}^{2C}(t-1)$, then the search is stopped
end while [new bounds of the interval are found]
end while [new lower and upper solutions are found]

Algorithm 4: Formation of the lower and upper subsets of complete crisp solutions
while [new complete crisp solutions are found] **do**
Search for the null solutions of the optimization problem (11)
$\boldsymbol{V}_0^1 = (\boldsymbol{W}_1^{1,0},\ldots,\boldsymbol{W}_N^{1,0})$ for the lower subset $\underline{D}_p$, where for all $w_{IK}^{1,0} = 1$, $K \le \hat{X}_I^p$
$\boldsymbol{V}_0^2 = (\boldsymbol{W}_1^{2,0},\ldots,\boldsymbol{W}_N^{2,0})$ for the upper subset $\overline{D}_p$, where for all $w_{IK}^{2,0} = 1$, $K \ge \hat{X}_I^p$
// Define the search space for the complete crisp solutions $\overleftarrow{\boldsymbol{V}}_{pl}$ and $\overrightarrow{\boldsymbol{V}}_{ph}$
$\overleftarrow{w}_{IK}^{pl} \in \{w_{IK}^{1,0},\ 1\}$, $\overrightarrow{w}_{IK}^{ph} \in \{w_{IK}^{2,0},\ 1\}$
Exclude previous solutions $\overleftarrow{\boldsymbol{V}}_{ps}$, $s < l$, and $\overrightarrow{\boldsymbol{V}}_{ps}$, $s < h$, from the search space
while [new weights are activated to cover the interval] **do** // Stepwise increment
Go to $\boldsymbol{V}^1(t) = (\boldsymbol{W}_1^1(t),\ldots,\boldsymbol{W}_N^1(t))$, $w_{IK}^1(t) \ge w_{IK}^1(t-1)$, $K \le \hat{X}_I^p$
Go to $\boldsymbol{V}^2(t) = (\boldsymbol{W}_1^2(t),\ldots,\boldsymbol{W}_N^2(t))$, $w_{IK}^2(t) \ge w_{IK}^2(t-1)$, $K \ge \hat{X}_I^p$
if $\Delta(\boldsymbol{V}^1(t)) = \Delta(\boldsymbol{V}_0^1)$, then $\boldsymbol{V}^1 \in \underline{D}_p$
if $\Delta(\boldsymbol{V}^2(t)) = \Delta(\boldsymbol{V}_0^2)$, then $\boldsymbol{V}^2 \in \overline{D}_p$
// Define the lower (upper) bounded complete crisp solution
if $\boldsymbol{V}^1(t) \ne \boldsymbol{V}^1(t-1)$, then $\overleftarrow{w}_{IK}^{pl} = w_{IK}^1(t)$ for the lower subset $\underline{D}_p$
if $\boldsymbol{V}^2(t) \ne \boldsymbol{V}^2(t-1)$, then $\overrightarrow{w}_{IK}^{ph} = w_{IK}^2(t)$ for the upper subset $\overline{D}_p$
if $\boldsymbol{V}^1(t) = \boldsymbol{V}^1(t-1)$ and $\boldsymbol{V}^2(t) = \boldsymbol{V}^2(t-1)$, then the search is stopped
end while [new weights are activated to cover the interval]
end while [new complete crisp solutions are found]

5 Example: Piston Pump Diagnostics

The aim of this section is to check the correctness of the inverse inference for a set of experimental data. Let us consider the diagnostics of faults causes of the piston pump, the functioning of which is determined by the cycles of discharge and suction.

Input parameters are: x_1 (x_4) – hydraulic resistance of the discharging (suction) main, $x_1 = [1.4,\ 3.2]$ ($x_4 = [1.7,\ 3.8]$) kg/cm^2; x_2 (x_5) – discharge (suction) valve clearance, $x_2 = \pm[0.1,\ 0.3]$ ($x_5 = \pm[0.1,\ 0.5]$) mm; x_3 (x_6) – leakage of the discharging (suction) main, $x_3 = [0.5, 2.1]$ ($x_6 = [1.0, 3.2]$) cm^2/min. The faults causes to be identified: c_{11} (c_{41}) – increase of resistance x_1 (x_4); c_{21} (c_{51}) – decrease of the clearance x_2 (x_5); c_{22} (c_{52}) – increase of the clearance x_2 (x_5); c_{31} (c_{61}) – increase of leakage x_3 (x_6). Output parameters are: y_1 – productivity, $y_1 = [15, 30]$ m^3/h; y_2 – force main pressure, $y_2 = [10, 20]$ kg/cm^2; y_3 – consumed power, $y_3 = [15, 24]$ kw. The observed effects are: e_{11} – productivity y_1 fall; e_{21} (e_{22}) – force main pressure y_2 drop (rise); e_{31} (e_{32}) – consumed power y_3 drop (rise).

For the fuzzy causes and effects, we use the bell-shaped membership function of the variable *u* to the term *T* (Rakityanskaya and Rotshtein 2007):

$$\mu^T(u) = \frac{1}{1+\left(\frac{u-\beta}{\sigma}\right)^2},$$

where β is the coordinate of the function maximum; σ is the concentration parameter.

To tune the fuzzy model, we used the measurements results for 290 pumps with different measures of faults significances. Following (Rakityanskaya and Rotshtein 2007, Rotshtein and Rakytyanska 2009), the essence of tuning consists in finding the parameters of causes and effects membership functions and the fuzzy relations, which minimize the root mean-squared error:

$$RMSE = \sqrt{\frac{1}{290}\sum_{k=1}^{290}\sum_{J=1}^{M}\left[\mu^{E_J}(y_j^k) - \mu^{E_J}(\mu^{C_1}(x_1^k),\ldots,\mu^{C_N}(x_n^k))\right]^2},$$

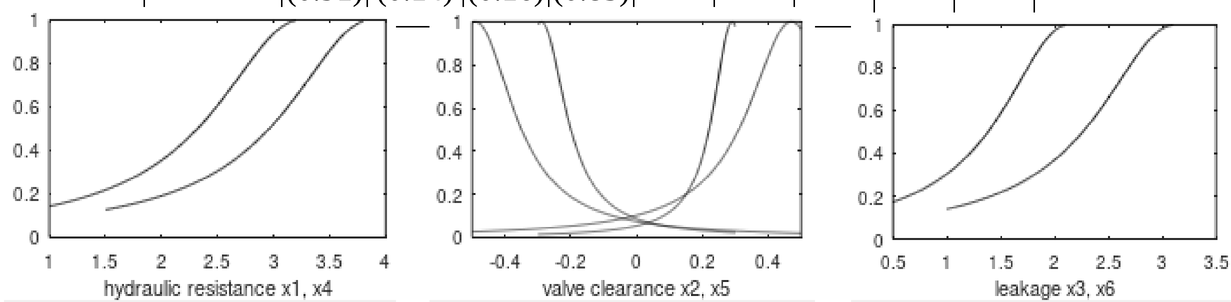
where x_i^k, y_j^k are the values of the input and output parameters in the *k*-th experiment.

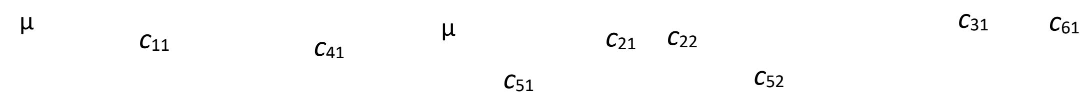
As a result of tuning, the *RMSE* takes the value of 0.0118.

Parameters of membership functions for the fuzzy causes and effects are given in Table 1. Membership functions are presented in Figure 1.

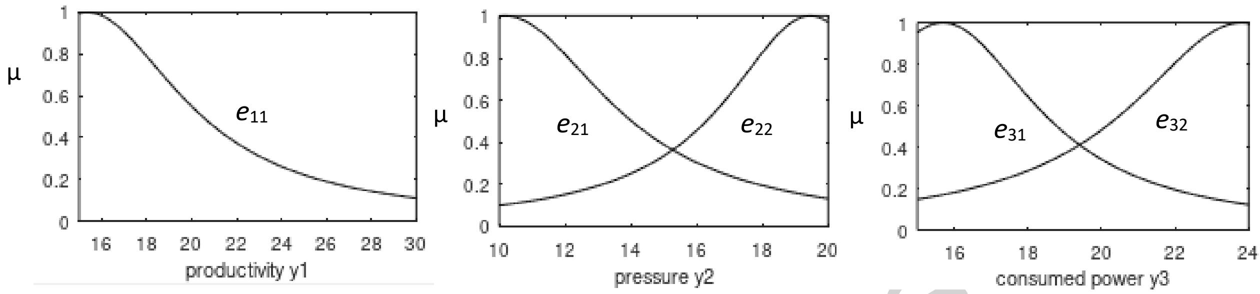
Table 1 – Membership functions parameters for the fuzzy causes and effects

Parameter	Fuzzy causes				Fuzzy effects				
	c_{11} (c_{41})	c_{21} (c_{51})	c_{22} (c_{52})	c_{31} (c_{61})	e_{11}	e_{21}	e_{22}	e_{31}	e_{32}
β	3.24 (3.87)	-0.29 (-0.49)	0.29 (0.47)	2.12 (3.11)	15.35	10.18	19.45	15.69	23.81
σ	0.92 (0.91)	0.09 (0.14)	0.07 (0.16)	0.74 (0.85)	5.14	3.85	3.16	3.10	3.68





a)



b)

Figure 1 – Membership functions of the fuzzy causes (a) and effects (b) after tuning

For aggregating the subsystems of discharging and suction, the cause-effect interconnections were modeled using the extended *max-min* inference rule (Yager and Filev 1994).

Let us redenote:

$(\mu^{C_1},\dots,\mu^{C_4}) = (\mu^{c_{11}},\mu^{c_{21}},\mu^{c_{22}},\mu^{c_{31}})$ and $(\mu^{C_5},\dots,\mu^{C_8}) = (\mu^{c_{41}},\mu^{c_{51}},\mu^{c_{52}},\mu^{c_{61}})$ are the fuzzy causes vectors for the subsystems of discharging and suction;

$(\mu^{E_1},\dots,\mu^{E_5}) = (\mu^{e_{11}},\mu^{e_{21}},\mu^{e_{22}},\mu^{e_{31}},\mu^{e_{32}})$ is the fuzzy effects vector.

The extended *max-min* SFRE can be represented in the form of the *max-min* subsystems, which are aggregated using the *min* operator:

$$\mu^{E_1} = [(\mu^{C_3} \wedge 0.92) \vee (\mu^{C_4} \wedge 0.79)] \wedge$$
$$\wedge [(\mu^{C_5} \wedge 0.32) \vee (\mu^{C_6} \wedge 0.86) \vee (\mu^{C_7} \wedge 0.29) \vee (\mu^{C_8} \wedge 0.55)],$$
$$\mu^{E_2} = [(\mu^{C_1} \wedge 0.65) \vee (\mu^{C_2} \wedge 0.43) \vee (\mu^{C_3} \wedge 0.57)] \wedge$$
$$\wedge [(\mu^{C_5} \wedge 0.53) \vee (\mu^{C_6} \wedge 0.74) \vee (\mu^{C_7} \wedge 0.39) \vee (\mu^{C_8} \wedge 0.46)],$$
$$\mu^{E_3} = [\mu^{C_2} \wedge 0.89] \wedge [(\mu^{C_7} \wedge 0.78) \vee (\mu^{C_8} \wedge 0.91)],$$
$$\mu^{E_4} = [(\mu^{C_1} \wedge 0.82) \vee (\mu^{C_3} \wedge 0.71) \vee (\mu^{C_4} \wedge 0.35)] \wedge [(\mu^{C_5} \wedge 0.76) \vee (\mu^{C_6} \wedge 0.41)],$$
$$\mu^{E_5} = [(\mu^{C_1} \wedge 0.47) \vee (\mu^{C_2} \wedge 0.96)] \wedge [(\mu^{C_5} \wedge 0.38) \vee (\mu^{C_7} \wedge 0.90)]. \quad (13)$$

Using the distributive set theoretic law, the extended *max-min* SFRE (13) can be rearranged in the form of the dual *min-max* subsystems, which are aggregated using the *max* operator:

$$\mu^{E_1} = (\mu^{H_9} \wedge 0.32) \vee (\mu^{H_{10}} \wedge 0.86) \vee (\mu^{H_{11}} \wedge 0.29) \vee (\mu^{H_{12}} \wedge 0.55) \vee$$
$$\vee (\mu^{H_{13}} \wedge 0.32) \vee (\mu^{H_{14}} \wedge 0.79) \vee (\mu^{H_{15}} \wedge 0.29) \vee (\mu^{H_{16}} \wedge 0.55),$$
$$\mu^{E_2} = (\mu^{H_1} \wedge 0.53) \vee (\mu^{H_2} \wedge 0.65) \vee (\mu^{H_3} \wedge 0.39) \vee (\mu^{H_4} \wedge 0.46) \vee$$
$$\vee (\mu^{H_5} \wedge 0.43) \vee (\mu^{H_6} \wedge 0.43) \vee (\mu^{H_7} \wedge 0.39) \vee (\mu^{H_8} \wedge 0.43) \vee$$
$$\vee (\mu^{H_9} \wedge 0.53) \vee (\mu^{H_{10}} \wedge 0.57) \vee (\mu^{H_{11}} \wedge 0.39) \vee (\mu^{H_{12}} \wedge 0.46),$$
$$\mu^{E_3} = (\mu^{H_7} \wedge 0.78) \vee (\mu^{H_8} \wedge 0.89),$$
$$\mu^{E_4} = (\mu^{H_1} \wedge 0.76) \vee (\mu^{H_2} \wedge 0.41) \vee (\mu^{H_9} \wedge 0.71) \vee$$
$$\vee (\mu^{H_{10}} \wedge 0.41) \vee (\mu^{H_{13}} \wedge 0.35) \vee (\mu^{H_{14}} \wedge 0.35),$$
$$\mu^{E_5} = (\mu^{H_1} \wedge 0.38) \vee (\mu^{H_3} \wedge 0.47) \vee (\mu^{H_5} \wedge 0.38) \vee (\mu^{H_7} \wedge 0.90), \qquad (14)$$

where

$$\mu^{H_L} = \mu^{C_I} \wedge \mu^{C_K}, I = 1, \dots, 4, K = 5, \dots, 8, L = 1, \dots, 16.$$

Let us represent the observed parameters for a specific pump: y_1=19.12 m^3/h; y_2=13.37 kg/cm^2; y_3=18.90 kw. The measures of effects significances can be defined with the help of the membership functions in Fig. 1,b:

$$\mu^{e_{11}}(y_1) = 0.65;\ \mu^{e_{21}}(y_2) = 0.59;\ \mu^{e_{22}}(y_2) = 0.21;\ \mu^{e_{31}}(y_3) = 0.48;\ \mu^{e_{32}}(y_3) = 0.36.$$

The set of interval solutions of the SFRE (13) is defined as follows (Tables 2-6). The set of aggregating solutions $\hat{G}$ is presented in Table 2. The number of aggregating solutions is $\hat{Z}$=3. When finding the lower and upper subsets $\underline{S}_p$ and $\overline{S}_p$ for each aggregating solution $\hat{\boldsymbol{\mu}}_p^C$, the measures of effects significances $\mu_i^{E_J}(\hat{\boldsymbol{\mu}}_p^C)$ in the *max-min* subsystems of the SFRE (13) are presented in Table 3, and the measures of causes combinations significances $\mu_J^{H_L}(\hat{\boldsymbol{\mu}}_p^C)$ in the dual *min-max* subsystems of the SFRE (14) are presented in Table 4.

The set of minimal solutions $\underline{B}_p$ and the set of maximal solutions $\overline{B}_p$ for each aggregating solution $\hat{\boldsymbol{\mu}}_p^C \in \hat{G}$ are presented in Tables 5, 6. The number of minimal solutions is: $\underline{Z}_1 = 4$ for $\hat{\boldsymbol{\mu}}_1^C$; $\underline{Z}_2 = 8$ for $\hat{\boldsymbol{\mu}}_2^C$; $\underline{Z}_3 = 4$ for $\hat{\boldsymbol{\mu}}_3^C$. The number of maximal solutions is: $\overline{Z}_1 = 4$ for $\hat{\boldsymbol{\mu}}_1^C$; $\overline{Z}_2 = 2$ for $\hat{\boldsymbol{\mu}}_2^C$; $\overline{Z}_3 = 2$ for $\hat{\boldsymbol{\mu}}_3^C$.

The total number of interval solutions is $Z = 40$, since $\underline{Z}_1 \cdot \overline{Z}_1 = 16$ for $\hat{\boldsymbol{\mu}}_1^C$; $\underline{Z}_2 \cdot \overline{Z}_2 = 16$ for $\hat{\boldsymbol{\mu}}_2^C$; $\underline{Z}_3 \cdot \overline{Z}_3 = 8$ for $\hat{\boldsymbol{\mu}}_3^C$. For each interval solution, the optimization criterion (9) takes the value of $\Delta(\boldsymbol{\mu}^C) = 0.0004$.

Table 2 – Set of aggregating solutions

№	Causes significance measures							
	Discharging subsystem				Suction subsystem			
	$\hat{\mu}^{C_1}$	$\hat{\mu}^{C_2}$	$\hat{\mu}^{C_3}$	$\hat{\mu}^{C_4}$	$\hat{\mu}^{C_5}$	$\hat{\mu}^{C_6}$	$\hat{\mu}^{C_7}$	$\hat{\mu}^{C_8}$
1	0.59	0.21	0.65	0.65	0.48	0.65	0.21	0.21
2	0.36	0.36	0.65	0.65	0.48	0.65	0.21	0.21
3	0.36	0.21	0.65	0.65	0.48	0.65	0.21	0.21

Table 3 – Measures of effects significances for lower subsets of interval solutions

max-min subsystem	Fuzzy effects				
	J=1	*J*=2	*J*=3	*J*=4	*J*=5
$\mu_1^{E_J}$	0.65	0.59/0.57/0.57	0.21/0.36/0.21	0.65	0.47/0.36/0.36
$\mu_2^{E_J}$	0.65	0.65	0.21	0.48	0.38
$\wedge\,[\mu_i^{E_J}]$	0.65	0.59/0.57/0.57	0.21	0.48	0.38/0.36/0.36

Table 4 – Measures of causes combinations significances for upper subsets of interval solutions

min-max subsystem	Fuzzy effects				
	J=1	*J*=2	*J*=3	*J*=4	*J*=5
μ^{H_1}	-	0.48/0.36/0.36	-	0.48/0.36/0.36	0.38/0.36/0.36
μ^{H_2}	-	0.59/0.36/0.36	-	0.41/0.36/0.36	-
μ^{H_5}	-	0.21/0.36/0.21	-	-	0.21/0.36/0.21
μ^{H_7}	-	0.21	0.21	-	0.21
μ^{H_8}	-	0.21	0.21	-	-
μ^{H_9}	0.32	0.48	-	0.48	-
$\mu^{H_{10}}$	0.65	0.57	-	0.41	-
$\mu^{H_{14}}$	0.65	-	-	0.35	-
$\vee\,[\mu_J^{H_L}]$	0.65	0.59/0.57/0.57	0.21	0.48	0.38/0.36/0.36

Table 5 – Set of minimal solutions

№	Causes significance measures							
	Discharging subsystem				Suction subsystem			
	$\underline{\mu}^{C_1}$	$\underline{\mu}^{C_2}$	$\underline{\mu}^{C_3}$	$\underline{\mu}^{C_4}$	$\underline{\mu}^{C_5}$	$\underline{\mu}^{C_6}$	$\underline{\mu}^{C_7}$	$\underline{\mu}^{C_8}$
1,1 1,2	0.59	0.21	0.65 0	0 0.65	0.48	0.65	0.21	0
1,3 1,4	0.59	0.21	0.65 0	0 0.65	0.48	0.65	0	0.21
2,1 2,2	0.36	0.21	0.65 0.57	0 0.65	0.48	0.65	0.21	0
2,3 2,4	0.36	0.21	0.65 0.57	0 0.65	0.48	0.65	0	0.21
2,5 2,6	0	0.36	0.65 0.57	0 0.65	0.48	0.65	0.21	0
2,7 2,8	0	0.36	0.65 0.57	0 0.65	0.48	0.65	0	0.21
3,1 3,2	0.36	0.21	0.65 0.57	0 0.65	0.48	0.65	0.21	0
3,3 3,4	0.36	0.21	0.65 0.57	0 0.65	0.48	0.65	0	0.21

Table 6 – Set of maximal solutions

№	Causes significance measures							
	Discharging subsystem				Suction subsystem			
	$\overline{\mu}^{C_1}$	$\overline{\mu}^{C_2}$	$\overline{\mu}^{C_3}$	$\overline{\mu}^{C_4}$	$\overline{\mu}^{C_5}$	$\overline{\mu}^{C_6}$	$\overline{\mu}^{C_7}$	$\overline{\mu}^{C_8}$
1,1 1,2	0.59	1	0.65 1	0.65 1	0.48	1 0.65	0.21	0.21
1,3 1,4	0.59	0.21	0.65 1	0.65 1	0.48	1 0.65	0.38	1
2,1 2,2	0.36	0.36	0.65 1	0.65 1	0.48	1 0.65	0.21	0.21
3,1 3,2	0.36	0.21	0.65 1	0.65 1	0.48	1 0.65	1	1

We shall describe the measures of causes significances μ^{C_I} by the following linguistic modifiers α_{IK}: *weak* (*w*), *moderate* (*m*), *essential* (*e*), *strong* (*s*) *decrease* (*D*) or *increase* (*I*). It is supposed that the upper bounds of input parameters for each modifier are known:

$$\overline{x}_1(wI) = 1.9;\ \overline{x}_1(mI) = 2.4;\ \overline{x}_1(eI) = 2.8;\ \overline{x}_1(sI) = 3.2;$$
$$\overline{x}_2(wI/wD) = \pm 0.15;\ \overline{x}_2(mI/mD) = \pm 0.2;\ \overline{x}_2(eI/eD) = \pm 0.25;\ \overline{x}_2(sI/sD) = \pm 0.3;$$
$$\overline{x}_3(wI) = 0.9;\ \overline{x}_3(mI) = 1.3;\ \overline{x}_3(eI) = 1.7;\ \overline{x}_3(sI) = 2.1;$$
$$\overline{x}_4(wI) = 2.4;\ \overline{x}_4(mI) = 3.0;\ \overline{x}_4(eI) = 3.4;\ \overline{x}_4(sI) = 3.8;$$
$$\overline{x}_5(wI/wD) = \pm 0.2;\ \overline{x}_5(mI/mD) = \pm 0.3;\ \overline{x}_5(eI/eD) = \pm 0.4;\ \overline{x}_5(sI/sD) = \pm 0.5;$$
$$\overline{x}_6(wI) = 1.6;\ \overline{x}_6(mI) = 2.1;\ \overline{x}_6(eI) = 2.6;\ \overline{x}_6(sI) = 3.2.$$

In this case, the constraints imposed on the measures of causes significances can be determined using the membership functions in Fig. 1,a:

$$\mu^{c_{11}}(x_1) \in \{0.32, 0.55, 0.81, 1\};$$
$$\mu^{c_{21}}(x_2) \in \{0.28, 0.50, 0.83, 1\};\ \mu^{c_{22}}(x_2) \in \{0.20, 0.38, 0.75, 1\};$$
$$\mu^{c_{31}}(x_3) \in \{0.27, 0.45, 0.69, 1\};$$
$$\mu^{c_{41}}(x_4) \in \{0.28, 0.52, 0.79, 1\};$$
$$\mu^{c_{51}}(x_5) \in \{0.19, 0.35, 0.71, 1\};\ \mu^{c_{52}}(x_5) \in \{0.26, 0.47, 0.84, 1\};$$
$$\mu^{c_{61}}(x_6) \in \{0.24, 0.41, 0.74, 1\}.$$

Under the given constraints, the set of linguistic solutions of the SFRE (13) is defined as follows (Tables 7-11). The set of constrained aggregating solutions $\widehat{D}$ is presented in Table 7. The number of constrained aggregating solutions is $\hat{Q} = 3$. When finding the lower and upper subsets $\underline{D}_p$ and $\overline{D}_p$ for each constrained aggregating solution $\widehat{\boldsymbol{V}}_p$, the measures of effects significances $\mu_i^{E_J}(\widehat{\boldsymbol{V}}_p)$ in the *max-min* subsystems of the SFRE (13) are presented in Table 8, and the measures of causes combinations significances $\mu_J^{H_L}(\widehat{\boldsymbol{V}}_p)$ in the dual *min-max* subsystems of the SFRE (14) are presented in Table 9.

The lower and upper subsets of complete crisp solutions $\underline{D}_p$ and $\overline{D}_p$ for each constrained aggregating solution $\widehat{\boldsymbol{V}}_p \in \widehat{D}$ are presented in Tables 10, 11. The number of upper bounded complete crisp solutions is: $\underline{Q}_1 = 2$ for $\widehat{\boldsymbol{V}}_1$; $\underline{Q}_2 = 1$ for $\widehat{\boldsymbol{V}}_2$; $\underline{Q}_3 = 1$ for $\widehat{\boldsymbol{V}}_3$. The number of lower bounded complete crisp solutions is: $\overline{Q}_1 = 2$ for $\widehat{\boldsymbol{V}}_1$; $\overline{Q}_2 = 2$ for $\widehat{\boldsymbol{V}}_2$; $\overline{Q}_3 = 1$ for $\widehat{\boldsymbol{V}}_3$.

The total number of constrained solutions is $Q = 7$, since $\underline{Q}_1 \cdot \overline{Q}_1 = 4$ for $\widehat{\boldsymbol{V}}_1$; $\underline{Q}_2 \cdot \overline{Q}_2 = 2$ for $\widehat{\boldsymbol{V}}_2$; $\underline{Q}_3 \cdot \overline{Q}_3 = 1$ for $\widehat{\boldsymbol{V}}_3$. For each constrained solution, the optimization criterion (11) does not exceed the value of $\Delta(\boldsymbol{V}) = 0.0125$, which allows an average absolute error of 0.05 for each equation in (13).

Table 7 – Set of constrained aggregating solutions

№	Weights of linguistic modifiers							
	Discharging subsystem				Suction subsystem			
	$\widehat{\boldsymbol{W}}_1$	$\widehat{\boldsymbol{W}}_2$	$\widehat{\boldsymbol{W}}_3$	$\widehat{\boldsymbol{W}}_4$	$\widehat{\boldsymbol{W}}_5$	$\widehat{\boldsymbol{W}}_6$	$\widehat{\boldsymbol{W}}_7$	$\widehat{\boldsymbol{W}}_8$
1	0100	1000	0010	0010	0100	0010	1000	1000
2	0100	1000	0100	0010	0100	0010	1000	1000
3	1000	1000	0010	0010	0100	0010	1000	1000

Table 8 – Measures of effects significances for lower subsets of constrained solutions

max-min subsystem	Fuzzy effects				
	J=1	*J*=2	*J*=3	*J*=4	*J*=5
$\mu_1^{E_J}$	0.75/0.69/0.75	0.57/0.55/0.57	0.28	0.71/0.55/0.71	0.47/0.47/0.32
$\mu_2^{E_J}$	0.71	0.71	0.26	0.52	0.38
$\wedge\,[\mu_i^{E_J}]$	0.71/0.69/0.71	0.57/0.55/0.57	0.26	0.52	0.38/0.38/0.32

Table 9 – Measures of causes combinations significances for upper subsets of constrained solutions

min-max subsystem	Fuzzy effects				
	J=1	*J*=2	*J*=3	*J*=4	*J*=5
μ^{H_1}	-	0.52/0.52/0.32	-	0.52/0.52/0.32	0.38/0.38/0.32
μ^{H_2}	-	0.55/0.55/0.32	-	0.41/0.41/0.32	-
μ^{H_5}	-	0.28	-	-	0.28
μ^{H_7}	-	0.26	0.26	-	0.26
μ^{H_8}	-	0.24	0.24	-	-
μ^{H_9}	0.32	0.52/0.38/0.52	-	0.52/0.38/0.52	-
$\mu^{H_{10}}$	0.71/0.38/0.71	0.57/0.38/0.57	-	0.41/0.38/0.41	-
$\mu^{H_{14}}$	0.69	-	-	0.35	-
$\vee\,[\mu_J^{H_L}]$	0.71/0.69/0.71	0.57/0.55/0.57	0.26	0.52	0.38/0.38/0.32

Table 10 – Lower subsets of complete crisp solutions

№	Weights of linguistic modifiers							
	Discharging subsystem				Suction subsystem			
	$\overleftarrow{\boldsymbol{W}}_1$	$\overleftarrow{\boldsymbol{W}}_2$	$\overleftarrow{\boldsymbol{W}}_3$	$\overleftarrow{\boldsymbol{W}}_4$	$\overleftarrow{\boldsymbol{W}}_5$	$\overleftarrow{\boldsymbol{W}}_6$	$\overleftarrow{\boldsymbol{W}}_7$	$\overleftarrow{\boldsymbol{W}}_8$
1,1 1,2	0100	1000	0010 1110	1110 0010	0100	0010	1000	1000
2,1	0100	1000	1100	0010	0100	0010	1000	1000
3,1	1000	1000	0010	1110	0100	0010	1000	1000

Table 11 – Upper subsets of complete crisp solutions

№	Weights of linguistic modifiers							
	Discharging subsystem				Suction subsystem			
	$\overrightarrow{\boldsymbol{W}}_1$	$\overrightarrow{\boldsymbol{W}}_2$	$\overrightarrow{\boldsymbol{W}}_3$	$\overrightarrow{\boldsymbol{W}}_4$	$\overrightarrow{\boldsymbol{W}}_5$	$\overrightarrow{\boldsymbol{W}}_6$	$\overrightarrow{\boldsymbol{W}}_7$	$\overrightarrow{\boldsymbol{W}}_8$
1,1 1,2	0100	1111 1000	0011	0011	0100	0010	1000	1000 1111
2,1 2,2	0100	1111 1000	0100	0010	0111 0100	0011	1000	1000 1111
3,1	1000	1000	0011	0011	0100	0010	1111	1111

Thus, for the observed state of the piston pump, the constrained solutions provide the linguistic interpretation of the interval solutions in the form of the set of explanations:

(x_1=*wI–mI* OR x_2=*eI–sI* OR x_3=*wI–sI*) AND (x_4=*mI* OR x_5=*eD* OR x_6=*wI–sI*);
(x_1= *mI* OR x_2=*wI–sI* OR x_3=*eI–sI*) AND (x_4=*mI* OR x_5=*eD* OR x_6=*wI–sI*);
(x_1= *mI* OR x_2=*wI–mI* OR x_3=*eI*) AND (x_4=*mI–sI* OR x_5=*eD–sD* OR x_6= *wI*);
(x_1= *mI* OR x_2= *wI–mI* OR x_3=*eI*) AND (x_4=*mI* OR x_5=*eD-sD* OR x_6=*wI-sI*).

For each significance level of faults, the accuracy characteristics of the genetic algorithm are given in Table 12 as the ratio of the number of correct diagnoses to the number of cases in the dataset. Correctness of diagnostics at the level of 95% can be attained while evaluating the faults significance measures using four levels described by linguistic modifiers.

Table 12 – Accuracy characteristics of the genetic algorithm

Significance level of faults	Probability of correct diagnosis							
	Discharging subsystem				Suction subsystem			
	c_{11}	c_{21}	c_{22}	c_{31}	c_{41}	c_{51}	c_{52}	c_{61}
wI (*wD*)	62/65 = 0.95	27/27 = 1.0	33/34 = 0.97	80/84 = 0.95	71/73 = 0.97	27/28 = 0.96	25/25 = 1.0	67/69 = 0.97
mI (*mD*)	75/78 = 0.96	34/35 = 0.97	47/49 = 0.96	76/79 = 0.96	78/82 = 0.95	39/41 = 0.95	35/36 = 0.97	82/85 = 0.96
eI (*eD*)	92/96 = 0.96	42/44 = 0.95	51/53 = 0.96	88/90 = 0.98	85/89 = 0.95	45/47 = 0.96	57/60 = 0.95	78/82 = 0.95
sI (*sD*)	50/51 = 0.98	19/20 = 0.95	28/28 = 1.0	37/37 = 1.0	44/46 = 0.96	23/24 = 0.96	28/29 = 0.97	53/54 = 0.98

6. Estimating the efficiency of genetic search for the set of constrained solutions

Following (Peeva and Kyosev 2004), the problem of finding the set of solutions of the *max-min* SFRE belongs to the class of NP-hard problems with time complexity $O(N!)$, and determination of the maximum number of minimal solutions is a still open combinatorial problem. Therefore, in (Rakityanskaya and Rotshtein 2007, Rotshtein and Rakytyanska 2009), the genetic algorithm for solving the *max-min* SFRE was proposed. This section compares performance estimates of genetic search for interval and constrained solutions of the extended SFRE. When finding the set of solutions, the genetic algorithm includes formation of the null solution, the single lower (upper) bound, the set of lower (upper) bounds (Rotshtein and Rakytyanska 2011, 2012).

For both interval and constrained solutions, the search for the lower and upper subsets is distributed between two computers of the local computing cluster. The quad-core (Intel Core i5-7400 3.0 Ghz) processor provides parallelization of the independent processes of finding the lower subsets $\underline{S}_p$ or $\underline{D}_p$ for each aggregating solution $\hat{\boldsymbol{\mu}}_p^C$, $p = 1, \ldots, \hat{Z}$, or $\widehat{\boldsymbol{V}}_p$, $p = 1, \ldots, \hat{Q}$. Similarly, the search for the upper subsets $\overline{S}_p$ or $\overline{D}_p$ is parallelized. For both lower and upper subsets, the number of independent processes is equal to the number of aggregating solutions $\hat{Z}$ or $\hat{Q}$.

When searching for a null solution, the chromosome can be significantly shortened. Let L_0 (L_0^c) be the chromosome length for the null ordinary (constrained) solution. To code the ordinary solutions $\mu^{C_I} \in [0,\ 1]$, $I = 1, \dots, N$, with an accuracy of 2 digits, L_0=7N. To code the constrained solutions $\boldsymbol{W}_I$, $I = 1, \dots, N$, at the $g_I = 4$ levels, L_0^c=2N.

When searching for a single lower (upper) solution, the stepwise procedure for covering intervals is simplified. Let T_1 (T_1^c) be the number of iterations of the incremental search for the interval (constrained) solution. When finding the single lower (upper) bound $\underline{\mu}^{C_I}$ ($\overline{\mu}^{C_I}$), $I = 1, \dots, N$, the number of iterations depends on the null solution and can reach the maximum value for the widest interval [0, 1]. To avoid omitting the minimal (maximal) solutions, the step is 0.01 to provide an accuracy of 2 digits, that is, for the widest interval T_1=100. When covering intervals by the complete crisp solutions $\overleftarrow{\boldsymbol{W}}_I$ ($\overrightarrow{\boldsymbol{W}}_I$), $I = 1, \dots, N$, the number of iterations does not exceed the number of linguistic modifiers, that is, $T_1 = g_I = 4$.

When searching for a set of bounded solutions, the repeated runs of the genetic algorithm are shortened due to reduction of the number of constrained solutions. Let $\hat{Z}_{max}$ ($\hat{Q}_{max}$) be the maximum number of ordinary (constrained) aggregating solutions; Z_{max} (Q_{max}) be the maximum number of lower or upper ordinary (constrained) solutions for each solution subset. Following (Bartl and Belohlavek 2015), we estimate the maximum number of aggregating and minimal (maximal) solutions for the following problem size: n=2, M=4…10, N=2M=8...20, where n is the number of *max-min* subsystems. Based on the assumption that the solution takes either the lower or the upper value, $Z_{max} = 2^{N/2}$ (Bartl and Belohlavek 2015). Similarly, if aggregation is provided by one of the subsystems, $\hat{Z}_{max} = n^M$.

Under constraints (6) imposed on the solution granularity, the probability of finding all the minimal (maximal) solutions is proportional to the chromosome length and is equal to L_0^c/L_0. Using the probabilistic approach to estimating the number of constrained solutions (Bartl and Trnecka 2021), for both minimal (maximal) and aggregating solutions Q_{max}=$Z_{max} \cdot (L_0^c/L_0)$ and $\hat{Q}_{max}$=$\hat{Z}_{max} \cdot (L_0^c/L_0)$. Thus, for the given problem size, the number of repeated runs of the genetic algorithm required to find the complete solution set is reduced from Z_{max}=$\hat{Z}_{max}$=2^4…2^{10}=16…1024 to Q_{max}=$\hat{Q}_{max}$=2^4·(2/7)…2^{10}·(2/7)=5…293 for each session of the pool of optimization problems (9) or (11).

Thus, the piston pump diagnostics requires multiple solving the optimization problem (9) or (11) with $\hat{Z}_{max}$=16 ($\hat{Q}_{max}$=5) aggregating solutions; Z_{max}=16 (Q_{max}=5) lower (upper) solutions inside $\hat{Z} = 3$ ($\hat{Q} = 3$) parallel independent processes. Generation of interval and constrained solutions using principles of parallel computing requires 12 min and 3 min, respectively, which reduces the diagnostic time by 75%.

7. Conclusions

Even though the methods for solving the *max-min* SFRE are well developed, the applications for multifactorial dependencies require solving the extended *max-min* SFRE. Besides, for practical purposes it is sufficient to represent solutions in the form of linguistic modifiers that resolve the problem of finding all minimal solutions and eliminate excessive granularity (Bartl et al. 2012; Bartl and Trnecka 2021). When investigating the properties of the solution set, new types of solutions are introduced. Aggregating solutions make it possible to decompose the set of solutions into the lower and upper subsets defined by the unique greatest (least) aggregating solution and a set of minimal (maximal) solutions. To ensure interpretability of the interval solutions, the granular structure of the solution set is replaced by the relational one in the form of complete crisp solutions, that is maximum solutions for the vectors of binary weights of the linguistic modifiers.

Finding the solution set is reduced to solving the optimization problem using the genetic algorithm. The properties of the solution set allow us to parallelize the genetic search for the lower and upper subsets separated by the aggregating solutions. When solving the pool of optimization problems, the imposed constraints make it possible to simplify the search for the null solution; the single interval solution; the set of interval solutions. The computation time is shortened due to reduction of the number of the constrained aggregating and minimal (maximal) solutions proportionally to the chromosome length. With a given number of linguistic modifiers, the time required to form intervals using the complete crisp solutions is sharply decreased. The accuracy of the inverse inference is ensured by tuning the fuzzy relation model using experimental data.

Further research is to develop an inverse inference method based on a modified fuzzy relation matrix. In this case, the cause-effect connections are given by the linguistic modifiers (Bartl et al. 2012; Bartl and Trnecka 2021), and the constraints are imposed on the values of fuzzy relations. In addition to solving the problem of solutions sensitivity to changes in the model parameters, this approach will also simplify the search for the solution set of the SFRE.

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Statements and Declarations

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